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ELEMENTS

OF

PROJECTIVE GEOME

BY

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AUTHOR'S PREFACE TO T FIRST EDITION*

Amplissima et pulcherrima scientia figurarum. At quam est m nomen Geometriæ!—NICOD FRISCHLINUS, *Dialog* I

Perspectivæ methodus, qua nec inter inventas nec inter inventu po compendiosior esse videtur —B PASCAL, *Lit ad Acad Paris*, 16

Da veniam scriptis, quorum non gloria nobis
Causa, sed utilitas officiumque fuit —OVID, *ex Pont*, iii 9

THIS book is not intended for those whose high i is to advance the progress of science, they would f nothing new, neither as regards principles, nor as methods The propositions are all old, in fact, not them owe their origin to mathematicians of the mo antiquity They may be traced back to EUCLID (28 APOLLONIUS of Perga (247 B C), to PAPPUS of Alexar century after Christ), to DESARGUES of Lyons (15 to PASCAL (1623–1662), to DE LA HIRE (1640–1 NEWTON (1642–1727), to MACLAURIN (1698–1746), LAMBERT (1728–1777), &c The theories and metho make of these propositions a homogeneous and ha whole it is usual to call *modern*, because they have covered or perfected by mathematicians of an age ours, such as CARNOT, BRIANCHON, PONCELET, MOBIUS, CHASLES, STAUDT, &c, whose works were publish earlier half of the present century

Various names have been given to this subject of are about to develop the fundamental principles

not to adopt that of *Higher Geometry* (*Géométrie supérieure*, *höhere Geometrie*), because that to which the title 'higher' at one time seemed appropriate, may to-day have become very elementary, nor that of *Modern Geometry* (*neuere Geometrie*) which in like manner expresses a merely relative idea, and is moreover open to the objection that although the method may be regarded as modern, yet the matter is to a great extent old. Nor does the title *Geometry of position* (*Geometrie der Lage*) as used by STAUDT* seem to me a suitable one, since it excludes the consideration of the metrical properties of figures. I have chosen the name of *Projective Geometry* †, as expressing the true nature of the methods, which are based essentially on central projection or perspective. And one reason which has determined this choice is that the great PONCELET, the chief creator of the modern methods, gave to his immortal book the title of *Traité des propriétés projectives des figures* (1822).

In developing the subject I have not followed exclusively any one author, but have borrowed from all what seemed useful for my purpose, that namely of writing a book which should be thoroughly elementary, and accessible even to those whose knowledge does not extend beyond the mere elements of ordinary geometry. I might, after the manner of STAUDT, have taken for granted no previous notions at all, but in that case my work would have become too extensive, and would no longer have been suitable for students who have read the usual elements of mathematics. Yet the whole of what such students have probably read is not necessary in order to understand my book, it is sufficient that they should know the chief propositions relating to the circle and to similar triangles.

It is, I think, desirable that theoretical instruction i

* Equivalent to the *Descriptive Geometry* of CAYLEY (*Sixth memoir on quantics*, Phil. Trans. of the Royal Society of London, 1859, p. 90). The name *Géométrie de position* as used by CARNOT corresponds to an idea quite different from that which I wished to express in the title of my book. I leave out of consideration other names, such as *Geométrie segmentaire* and *Organische Geometrie*, as referring to ideas which are too limited in my opinion.

geometry should have the help afforded it by the pi
constructing and drawing of figures. I have according
more stress on *descriptive* properties than on *metrical* one
have followed rather the methods of the *Geometrie der*
STAUDT than those of the *Géométrie supérieure* of CHA
It has not however been my wish entirely to exclude m
properties, for to do this would have been detrimei
other practical objects of teaching † I have therefor
duced into the book the important notion of the *anh*
ratio, which has enabled me, with the help of the few
mentioned propositions of the ordinary geometry, to es
easily the most useful metrical properties, which are
consequences of the projective properties, or are closely
to them

I have made use of *central projection* in order to es
the idea of *infinitely distant elements*, and, following the e
of STEINER and of STAUDT, I have placed the law of
quite at the beginning of the book, as being a logi
which arises immediately and naturally from the pos
of constructing space by taking either the point or the]
element The enunciations and proofs which corres]
one another by virtue of this law have often been pl
parallel columns, occasionally however this arrangem
been departed from, in order to give to students the
tunity of practising themselves in deducing from a t
its correlative Professor REYE remarks, with justice
preface to his book, that Geometry affords nothing so
to a beginner, nothing so likely to stimulate him to
work, as the principle of duality, and for this reas
very important to make him acquainted with it
as possible, and to accustom him to employ it wi
fidence

The masterly treatises of PONCELET, STEINER, CHAS

* Cf REYE, *Geometrie der Lage* (Hannover, 1866, 2nd edition, 187
the preface

† ZECH, *Die hohere Geometrie in ihrer Anwendung auf Kegelsc*

STAUDT * are those to which I must acknowledge myself most indebted, not only because all who devote themselves to Geometry commence with the study of these works, but also because I have taken from them, besides the substance of the methods, the proofs of many theorems and the solutions of many problems. But along with these I have had occasion also to consult the works of APOLLONIUS, PAPPUS, DESARGUES, DE LA HIRE, NEWTON, MACLAURIN, LAMBERT, CARNOT, BRIANCHON, MOBIUS, BELLAVITIS, &c, and the later ones of ZECH, GASKIN, WITZSCHEL, TOWNSEND, REYE, POUDRA, FIEDLER, &c.

In order not to increase the difficulties, already very considerable, of my undertaking, I have relieved myself from the responsibility of quoting in all cases the sources from which I have drawn, or the original discoverers of the various propositions or theories I trust then that I may be excused if sometimes the source quoted is not the original one †, or if occasionally the reference is found to be wanting entirely In giving references, my desire has been chiefly to call the attention of the student to the names of the great geometers he titles of their works, which have become classical ssociation with certain great theorems of the illustrious ames of EUCLID, APOLLONIUS, PAPPUS, DESARGUES, PASCAL, NEWTON, CARNOT, &c will not be without advantage in assisting the mind to retain the results themselves, and in exciting that scientific curiosity which so often contributes to enlarge our knowledge

Another object which I have had in view in giving references is to correct the first impressions of those to whom the name *Projective Geometry* has a suspicious air of novelty Such

* PONCELET *Traité des propriétés projectives des figures* (Paris, 1822) STEINER, *Systematische Entwickelung der Abhängigkeit geometrischer Gestalten von einander &c* Berlin, 1832) CHASLES, *Traité de Géométrie supérieure* (Paris, 1852) *Traité des sections coniques* (Paris, 1865) STAUDT *Geometrie der Lage* (Nürnberg 1847)

† In quoting an author I have almost always cited such of his treatises as are of considerable extent and generally known although his discoveries may have been originally announced elsewhere For example, the researches of CHASLES in the theory of conics date from a period in most cases anterior to the year 1830 those of STAUDT began in 1831, &c

persons I desire to convince that the subjects are to a gre extent of venerable antiquity, matured in the minds of t greatest thinkers, and now reduced to that form of extre simplicity which GERGONNE considered as the mark of perfecti in a scientific theory* In my analysis I shall follow t order in which the various subjects are arranged in the boo

The conception of *elements lying at an infinite distance* is d to the celebrated mathematician DESARGUES, who more th two centuries ago explicitly considered parallel straight lir as meeting in an infinitely distant point †, and parallel plar as passing through the same straight line at an infin distance ‡

The same idea was thrown into full light and ma generally known by PONCELET, who, starting from the post lates of the Euclidian Geometry, arrived at the conclusi that the points in space which lie at an infinite distance mi be regarded as all lying in the same plane §

DESARGUES || and NEWTON ¶ considered the asympto of the hyperbola as tangents whose points of contact lie at infinite distance

The name *homology* is due to PONCELET Homology, w reference to plane figures, is found in some of the earl treatises on perspective, for example in LAMBERT** or p haps even in DESARGUES ††, who enunciated and proved theorem concerning triangles and quadrilaterals in perspect or homology This theorem, for the particular case of t triangles (Art 17), is however really of much older date, a

* 'On ne peut se flatter d'avoir le dernier mot d une théorie, tant qu o peut pas l'expliquer en peu de paroles à un passant dans la rue (cf CHAS *Aperçu historique*, p 115)

† *Œuvres de* DESARGUES, *reunies et analysees par* M POUDRA (Paris, 18 tome 1 *Brouillon projet d une atteinte aux evénements des rencontres d'un avec un plan* (1639), pp 104, 105, 205

‡ *Loc cit*, pp 105, 106

§ *Traite des propriétés projectives des figures* (Paris, 1822) Arts 96, 580

|| *Loc cit*, p 210

¶ *Philosophiae naturalis principia mathematica* (1686), lib 1 prop scholium

** *Freie Perspective* 2nd edition (Zurich, 1774)

†† *Loc cit*, pp 413-416

is substantially identical with a celebrated porism of EUCLID (Art. 114), which has been handed down to us by PAPPUS * Homological figures in space were first studied by PONCELET †

The law of duality, as an independent principle, was enunciated by GERGONNE ‡, as a consequence of the theory of reciprocal polars (under the name *principe de réciprocité polaire*) it is due to PONCELET §

The geometric forms (range of points, flat pencil) are found, the names excepted, in DESARGUES and the later geometers STEINER || has defined them in a more explicit manner than any previous writer

The complete quadrilateral was considered by CARNOT ¶, the idea was extended by STEINER ** to polygons of any number of sides and to figures in space

Harmonic section was known to geometers of the most remote antiquity, the fundamental properties of it are to be found for example in APOLLONIUS †† DE LA HIRE ‡‡ gave the construction of the fourth element of a harmonic system by means of the harmonic property of the quadrilateral, *i e* by help of the ruler only

From 1832 the construction of projective forms was taught by STEINER §§

The complete theory of the anharmonic ratios is due to MOBIUS ||||, but before him EUCLID, PAPPUS ¶¶, DESARGUES ***, and BRIANCHON ††† had demonstrated the fundamental proposition of Art 63 DESARGUES ‡‡‡ was the author of the theory

* CHASLES, *Les trois livres de porismes d'Euclide, &c* (Paris, 1860), p 102

† *Loc cit*, pp 369 sqq

‡ *Annales de Mathematiques* vol xvi (Montpellier 1826), p 209

§ Ibid, vol viii (Montpellier, 1818), p 201

|| *Systematische Entwickelung*, pp xiii, xiv Collected Works, vol i p 237

¶ *De la correlation des figures de Geometrie* (Paris, 1801), p 122

** *Loc cit* pp 72 235, §§ 19, 55

†† *Conicorum* lib i 34 36, 37, 38

‡‡ *Sectiones conicae* (Parisiis, 1685), i 20

§§ *Loc cit* p 91

|||| *Der barycentrische Calcul* (Leipzig 1827), chap v

¶¶ *Mathematicae Collectiones*, vii 129

*** *Loc cit* p 425

††† *Memoire sur les lignes du second ordre* (Paris, 1817), p 7

‡‡‡ *Loc cit* pp 119 147 171 176

of *involution*, of which a few particular cases were al known to the Greek geometers *

The generation of conics by means of two projective 1 was set forth, forty years ago, by STEINER and by CHA it is based on two fundamental theorems (Arts. 149, from which the whole theory of these important curve be deduced The same method of generation include organic description of NEWTON † and various theore MACLAURIN

But the projectivity of the pencils formed by joining fixed points on a conic to a variable point on the sam already been proved, in other words, by APOLLONIUS ‡

When only sixteen years old (in 1640) PASCAL disco his celebrated theorem of the *mystic hexagram* §, and in BRIANCHON deduced the correlative theorem (Art 15 means of the theory of pole and polar

The properties of the quadrilateral formed by four tan to a conic and of the quadrangle formed by their poi contact are to be found in the Latin appendix (*De rum geometricarum proprietatibus generalibus tractatus*) t *Algebra* of MACLAURIN, a posthumous work (London, He deduced from these properties methods for the struction of a conic by points or by tangents in several where five elements (points or tangents) are given problem, in its full generality, was solved at a later da BRIANCHON

The idea of considering two projective ranges of poi the same conic was explicitly set forth by BELLAVITIS ||

To CARNOT ¶ we owe a celebrated theorem (Art 385 cerning the segments which a conic determines on the si

* PAPPUS, *Mathematicae Collectiones*, lib vii props 37-56, 127, 128, 1

† *Loc cit* lib i lemma xxi

‡ *Conicorum* lib iii 54, 5 , 56 I owe this remark to Prof ZEUTHEN

§ *Letter of* LEIBNITZ *to* M PERIER in the *Œuvres de B Pascal* (E edition, vol v p 459)

|| *Saggio di geometria derivata* (Nuovi Saggi dell Accademia di Padova 1838), p 2 0, note

¶ *Géometr e d* … (Pa… 1803) Art 3[illegible]0

a triangle. Of this theorem also certain particular cases were known long before.*

In the *Freie Perspective* of LAMBERT we meet with elegant constructions for the solution of several problems of the first and second degrees by means of the ruler, assuming however that certain elements are given; but the possibility of solving all problems of the second degree by means of the ruler and a fixed circle was made clear by PONCELET; afterwards STEINER, in a most valuable little book, showed the manner of practically carrying this out (Arts. 238 sqq.).

The theory of pole and polar was already contained, under various names, in the works already quoted of DESARGUES † and DE LA HIRE ‡; it was perfected by MONGE §, BRIANCHON ‖, and PONCELET. The last-mentioned geometer derived from it the theory of polar reciprocation, which is essentially the same thing as the law of duality, called by him the 'principe de réciprocité polaire.'

The principal properties of *conjugate diameters* were expounded by APOLLONIUS in books ii and vii of his work on the Conics.

And lastly, the fundamental theorems concerning *foci* are to be found in book iii of APOLLONIUS, in book vii of PAPPUS, and in book viii of DE LA HIRE.

Those who desire to acquire a more extended and detailed knowledge of the progress of Geometry from its beginnings until the year 1830 (which is sufficient for what is contained in this book) have only to read that classical work, the *Aperçu historique* of CHASLES.

* APOLLONIUS, *Conicorum* lib. iii. 16–23. DESARGUES, *loc. cit.*, p. 202. DE LA HIRE *loc. cit.* book v. props. 10, 12. NEWTON *Enumeratio linearum tertii ordinis* (*Opticks*, London, 1704) p. 142.

† *Loc. cit.*, pp. 164. 186, 190 sqq.

‡ *Loc. cit.*, i. 21–28. ii. 23–30.

§ *Geometrie descriptive* (Paris, 1795) Art. 40.

‖ *Journal de l'École Polytechnique*, cahier xiii. (Paris 1806).

AUTHOR'S PREFACE TO TH

ENGLISH EDITION

IN April last year, when I was in Edinburgh on the of the celebration of the tercentenary festival of the Ur there, Professor SYLVESTER did me the honour of sayin his opinion a translation of my book on the Elements o tive Geometry might be useful to students at the Engl versities as an introduction to the modern geometrical The same favourable judgement was shown to me l mathematicians, especially in Oxford, which place I v the following month of May at the invitation of Profes VESTER There Professor PRICE proposed to me that assist in an English translation of my book, to be car by Mr C LEUDESDORF, Fellow of Pembroke College, published by the Clarendon Press I accepted the with pleasure, and for this reason In my opinion the excel in the art of writing text-books for mathematic ing, as regards the clear exposition of theories abundance of excellent examples, carefully selected, books exist in other countries which can compete w of SALMON and many other distinguished English aut could be named I felt it therefore to be a great hon my book should be considered by such competen worthy to be introduced into their colleges

Unless I am mistaken, the preference given to my over the many treatises on modern geometry publish Continent is to be attributed to the circumstance th have striven, to the best of my ability, to imitate th

models My intention was not to produce a book of high theories which should be of interest to the advanced mathematician, but to construct an elementary text-book of modest dimensions, intelligible to a student whose knowledge need not extend further than the first books of Euclid. I aimed therefore at simplicity and clearness of exposition, and I was careful to supply an abundance of examples of a kind suitable to encourage the beginner, to make him seize the spirit of the methods, and to render him capable of employing them

My book has, I think, done some service in Italy by helping to spread a knowledge of projective geometry, and I am encouraged to believe that it has not been unproductive of results even elsewhere, since I have had the honour of seeing it translated into French and into German

If the present edition be compared with the preceding ones, it will be seen that the book has been considerably enlarged and amended. All the improvements which are to be found in the French and the German editions have been incorporated, a new Chapter, on Foci, has been added, and every Chapter has received modifications, additions, and elucidations, due in part to myself, and in part to the translator

In conclusion, I beg leave to express my thanks to the eminent mathematician, the Savilian Professor of Geometry, who advised this translation, to the Delegates of the Clarendon Press, who undertook its publication, and to Mr Leudesdorf, who has executed it with scrupulous fidelity

L CREMONA

Rome, May 1885

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ELEMENTS OF PROJECTIVE GEOMETI

CHAPTER I

DEFINITIONS.

1 By a *figure* is meant any assemblage of points, str lines, and planes, the straight lines and planes are all considered as extending to infinity, without regard tc limited portions of space which are enclosed by them the word *triangle*, for example, is to be understood a sy consisting of three points and three straight lines conne these points two and two, a *tetrahedron* is a system consi of four planes and the four points in which these planes i sect three and three, &c

In order to secure uniformity of notation, we shall always d points by the capital letters A, B, C, , straight lines by the letters a, b, c, , planes by the Greek letters α, β, γ, Mor AB will denote that part of the straight line joining A and B is comprised between the points A and B, Aa will denote the which passes through the point A and the straight line a, a point common to the straight line a and the plane α, $\alpha\beta$ the str line formed by the intersection of the planes α, β, ABC the pl the three points A, B, C, $\alpha\beta\gamma$ the point common to the three α, β, γ, αBC the point common to the plane α and the straigl BC, $A\beta\gamma$ the plane passing through the point A and the tı line $\beta\gamma$, αBc the straight line common to the plane α and the Bc, $A\beta c$ the straight line joining the point A to the point The notation $\alpha BC \equiv A'$ we shall use to express that the point co to the plane α and the straight line BC coincides with the poı

$u \equiv ABC$ will express that the straight line u contains the points A, B, C, &c

2 *To project from a fixed point S* (the *centre of projection*) a figure ($ABCD$, $abcd$) composed of points and straight lines, is to construct the straight lines or *projecting rays* SA , SB , SC , SD , and the planes (*projecting planes*) Sa , Sb , Sc , Sd , We thus obtain a new figure composed of straight lines and planes which all pass through the centre S

3 *To cut by a fixed plane σ* (*transversal plane*) a figure ($\alpha\beta\gamma\delta$, $abcd$) made up of planes and straight lines, is to construct the straight lines or *traces* σa , $\sigma\beta$, $\sigma\gamma$, and the points or *traces* σa , σb , σc , By this means we obtain a new figure composed of straight lines and points lying in the plane σ

4\. *To project from a fixed straight line s* (the *axis*) a figure $ABCD$ composed of points, is to construct the planes sA, sB, sC, . The figure thus obtained is composed of planes which all pass through the axis s

5 *To cut by a fixed straight line s* (a *transversal*) a figure $\alpha\beta\gamma\delta$ composed of planes, is to construct the points $s\alpha$, $s\beta$, $s\gamma$, In this way a new figure is obtained, composed of points all lying on the fixed transversal s

6 If a figure is composed of straight lines a, b, c, which all pass through a fixed point or *centre* S, it can be *projected* from a straight line or *axis* s passing through S , the result is a figure composed of planes sa , sb , sc ,

7 If a figure is composed of straight lines a , b , c , all lying in a fixed plane, it may be cut by a straight line (transversal) s lying in the same plane , the figure which results is formed by the points sa , sb , sc , *

* The operations of projecting and cutting (*projection* and *section*) are the two fundamental ones of the Projective Geometry

CHAPTER II

CENTRAL PROJECTION, FIGURES IN PERSPECTIVE.

8 CONSIDER a plane figure made up of points A, B, C, straight lines $AB, AC,$, BC, Project these from a S not lying in the plane (σ) of the figure, and cut the SA, SB, SC, and the planes $SAB, SAC,$, SBC, by a versal plane σ' (Fig 1) The traces on the plane σ' projecting rays and planes will form a second figure, a picture of the first When we carry out the two operations by which this second figure is derived from the first, we are said *to project from a centre* (or *vertex*) S *a given figure* σ *upon a plane of projection* σ' The new figure σ' is called the *perspective image* or the *central projection* of the

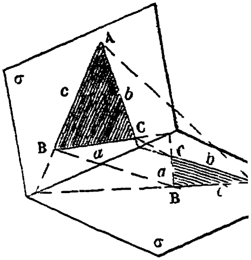


Fig 1

original one Of course, if the second figure be pro back from the centre S upon the plane σ, the first figur be formed again, *i e* the first figure is the projection second from the centre S upon the picture-plane σ Th figures σ and σ' are said to be *in perspective position*, or *in perspective*

9 If $A', B', C',$ are the traces of the rays SA, SB SC the plane σ', we may say that to the points A, B, C first figure *correspond* the points $A', B', C',$ of the s with the condition that two corresponding points alwa on a straight line passing through S If the point A de a straight line a in the plane σ, the ray SA will desc plane Sa, and therefore A' will describe a straight line intersection of the planes Sa and σ' The straight lines a

in which the planes σ and σ' are cut by any the same projecting plane, may thus be called *corresponding lines* It follows from this that to the straight lines $AB, AC, \ldots, BC, \ldots$ correspond the straight lines $A'B', A'C', \ldots, B'C', \ldots$ and that to all straight lines which pass through a given point A of the plane σ correspond straight lines which pass through the corresponding point A' of the plane σ'

10 If the point A describe a curve in the plane σ, the corresponding point A' will describe another curve in the plane σ', which may be said to *correspond* to the first curve Tangents to the two curves at corresponding points are clearly corresponding straight lines, and again, the two curves are cut by corresponding straight lines in corresponding points Two corresponding curves are therefore of the same degree *.

11. The two figures may equally well be generated by the simultaneous motion of a pair of corresponding straight lines a, a' If a revolve about a fixed point A, then a' will always pass through the corresponding point A'

Similarly, if a envelop a curve, then a' will envelop the corresponding curve The lines a and a', in corresponding positions, touch the two curves at corresponding points, and again, to the tangents to the first curve from a point A correspond the tangents to the second from the corresponding point A' Two corresponding curves are therefore of the same class †

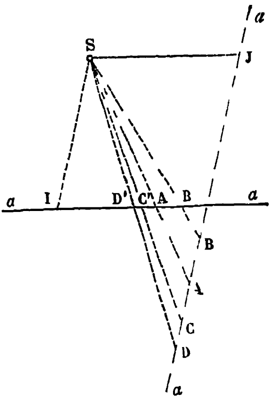


Fig 2

12 Consider two straight lines a and a' which correspond to one another in the figures σ, σ' (Fig 2) Every ray drawn through S in their plane meets them in two points, say A and A', which correspond to one another If the ray change its position and revolve round S, the points A and A' change their positions simultaneously, when the ray is about to

* The *degree* of a curve is the greatest number of points in which it can be cut by any arbitrary plane In the case of a plane curve it is the greatest number of points in which it can be cut by any straight line in the plane

† The *class* of a plane curve is the greatest number of tangents which can be drawn to it from any arbitrary point in the plane

become parallel to a, the point A' approaches I' (the where a' is cut by the straight line drawn through S para a) and the point A moves away indefinitely In order th property that to one point of a' corresponds one poin may hold universally, we say that the line a has a *po infinity* I, with which the point A coincides when A' coi with I', viz when the ray, turning about S, becomes pa to a The straight line a has only one point at infin being assumed that we can draw through S only on parallel to a *

The point I', the image of the point at infinity I, is the *vanishing point* of a'

Similarly, the straight line a' has a point J' at in which corresponds to the point J where a is cut by th drawn through S parallel to a'

Two parallel straight lines have the same point at in All straight lines which are parallel to a given stra must be considered as having a common point of inte at infinity

Two straight lines lying in the same plane always int in a point (finite or infinitely distant)

13 If now the straight line a takes all possible positi the plane σ, the corresponding straight line a' will alwa determined by the intersection of the planes σ' and Sa moves, the ray SI traces out a plane π parallel to σ an point I' describes the straight line $\pi\sigma'$, which we may by ι' This straight line ι' is then such that to any point on it corresponds a point at infinity in the plane σ, which belongs also to the plane π

We assume that the locus of these points at infinity plane σ is a straight line ι because it may be conside the intersection of the planes π and σ But this locus correspond to the straight line ι' in the plane σ', thus th that to every straight line in the plane σ' corresponds a st line in the plane σ holds without exception

The plane σ has only one straight line at infinity b through the point S only one plane parallel to σ can be The straight line ι', the image of the straight line at in is called the *vanishing line* of σ' It is parallel to $\sigma\sigma'$

* [illegible]

In the same way, the plane σ' has a straight line at infinity which corresponds to the intersection of the plane σ with the plane π' drawn through S parallel to σ'

Two parallel planes have the same straight line at infinity in common All planes parallel to a given plane must be considered as passing through a fixed straight line at infinity

If a straight line is parallel to a plane, the straight line at infinity in the plane passes through the point at infinity on the line If two straight lines are parallel, they meet in the same point the straight line at infinity in their plane

Two planes always cut one another in a straight line (finite or infinitely distant)

A straight line and a plane (not containing the line) always intersect in a point (finite or infinitely distant)

Three planes which do not contain the same straight line have always a common point (finite or infinitely distant)

14. Theorem *If two plane figures ABC , $A'B'C'$, (Fig 1) lying in different planes σ and σ', are in perspective, i e if the rays AA', BB', CC', meet in a point O, then the corresponding straight lines AB and $A'B'$, AC and $A'C'$, , BC and $B'C'$, will cut one another in points lying on the same straight line, viz the intersection of the planes of the two figures*

It is to be shown that if M is a point lying on the ...ight line $\sigma\sigma'$, and if a straight line a, lying in the plane σ, ...es through M, then the corresponding straight line a' will also pass through M But this is evidently the case, since the two straight lines a and a' are the intersections of the same projecting plane with the two planes σ and σ', and consequently the three straight lines $\sigma\sigma'$, a, and a' meet in a point, viz that common to the three planes The straight line $\sigma\sigma'$ is the locus of the points which correspond to themselves in the two figures

The vanishing line i' in the plane σ' is parallel to the straight line $\sigma\sigma'$, since i' and the corresponding straight line i, which lies entirely at an infinite distance in the plane σ, must intersect one another on $\sigma\sigma'$ Similarly, the vanishing line j of the plane σ is parallel to $\sigma\sigma'$

If each of the figures is a triangle, the theorem reads as follows —

If two triangles ABC and $A'B'C'$, lying respectively in the

planes σ and σ', are such that the straight lines AA', BB meet in a point S, then the three pairs of correspo sides, BC and $B'C'$, CA and $C'A'$, AB and $A'B'$, interse points lying on the straight line $\sigma\sigma'$

15 Conversely, *if to the points A, B, C, , and to the st lines AB, AC, , BC, of a plane figure σ correspond sev the points A', B', C', and the straight lines $A'B'$, $A'C'$, , I of another plane figure σ'*, in such a way that the corresp lines AB and $A'B'$, AC and $A'C'$, , BC and $B'C'$, m points lying on the line of intersection ($\sigma\sigma'$), of the planes σ a then the two figures are in perspective*

For if S be the point which is common to the planes AB $A'B'$, AC $A'C'$, BC $B'C'$, the three AA', BB', CC' of the trihedral angle formed by the planes will meet in S Similarly, the three planes AB AD $A'D'$, BD $B'D'$ meet in a point which is common edges AA', BB', DD', and this point is again S, since th straight lines AA', BB' suffice to determine it Therefc the straight lines AA', BB', CC', DD' pass throug same point S, that is, the two given figures are in perspe and S is their centre of projection

If each of the figures is a triangle, we have the the If two triangles ABC and $A'B'C'$, lying respectively planes σ and σ', are such that the sides BC and B' and $C'A'$, AB and $A'B'$ intersect one another two an in points lying on the straight line $\sigma\sigma'$, then the straigh AA', BB', CC' meet in a point S

16 THEOREM *If two triangles $A_1B_1C_1$ and $A_2B_2C_2$, lying same plane, are such that the straight lines A_1A_2, B_1B_2, C_1C in the same point O, then the three points of intersection of th B_1C_1 and B_2C_2, C_1A_1 and C_2A_2, A_1B_1 and A_2B_2 lie on a s line* (Fig 3)

Through the point O which is common to the st lines A_1A_2, B_1B_2, C_1C_2, draw any straight line outside the σ, and in this straight line take two points S_1 and S I the triangle $A_1B_1C_1$ from S_1 and the triangle $A_2B_2C_2$ fr The points A_1, A_2, O, S_2, S_1 lie in the same plane, the S_1A_1 and S_2A_2 meet one another (in A suppose), sin S_1B_1 and S_2B_2 (in B suppose) and S_1C_1 and S_2C_2 (in C sup

* The planes σ and σ' are to be regarded a distinct from each othe

Thus the triangle ABC is in perspective both with $A_1B_1C_1$ and with $A_2B_2C_2$ The straight lines BC, B_1C_1, B_2C_2 intersect in pairs and therefore meet in one and the same point A_0* Similarly CA, C_1A_1, and A_2C_2 meet in a point B_0, and AB, B_1A_1, and A_2B_2 in a point C_0 The three points A_0, B_0, C_0 lie on the straight line which is common to the planes σ and ABC The theorem is therefore proved

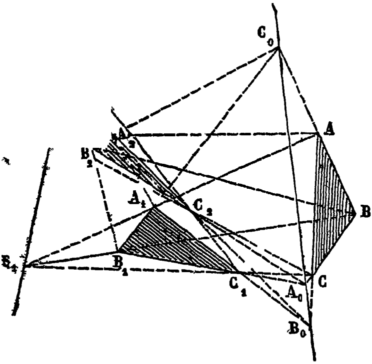


Fig 3

17 Conversely, *If two triangles $A_1B_1C_1$ and $A_2B_2C_2$, lying in the same plane, are such that the sides B_1C_1 and B_2C_2, C_1A_1 and C_2A_2, A_1B_1 and A_2B_2 cut one another in pairs in three collinear points A_0, B_0, C_0, then the straight lines A_1A_2, B_1B_2, C_1C_2, which join corresponding angular points, will pass through one and the same point O* (Fig 3)

Through the straight line $A_0B_0C_0$ draw another plane, and project, from an arbitrary centre S_1, the triangle $A_1B_1C_1$ upon this plane If ABC be the projection, the straight lines BC, B_1C_1 will cut one another in the point A_0, through which B_2C_2 will also pass, similarly AC will pass through B_0 and AB through C_0 The straight lines AA_2, BB_2, CC_2 intersect in pairs, without however all three lying in *the same* plane, they will therefore all meet in one point S_2 The straight lines S_1S_2 and A_1A_2 lie in the same plane, since S_1A_1 and S_2A_2 intersect in A, therefore S_1S_2 meets the three straight lines A_1A_2, B_1B_2, C_1C_2, i e A_1A_2, B_1B_2, C_1C_2 all meet in one point O, viz that which is common to the plane σ and the straight line S_1S_2 †

* BC is the intersection of the planes $S_1B_1C_1$ and $S_2B_2C_2$, which do not coincide, so that the straight lines BC, B_1C_1 and B_2C_2 do not all three lie in one plane The three planes BC B_1C_1, BC B_2C_2, and B_1C_1 B_2C_2 (or σ) intersect in the same point A_0

† Poncelet *Propr. proj. projectives des figures* (Paris 1822), Art 168 The theorems of Arts 11 and 12 are due to Desargues (*Œuvres*, ed Poudra, vol i p 413)

CHAPTER III

HOMOLOGY

Def. of homology and one method of constructing

18 CONSIDER a plane σ and another plane σ', in which l
lies any given figure made up of points and straight
Take two points S_1 and S_2 lying outside the given pl
and project from each of them as centre the given figure
to the plane σ In this way two new figures (σ_1 and σ_2
will be formed, which lie in the plane σ, and which ar
projections of one and the same figure σ' upon one an
same plane σ, but from different centres of projection.
two points A_1 and A_2, or two straight lines a_1 and a_2, i
figures σ_1 and σ_2 be said to *correspond* to each other
they are the images of one and the same point A'
one and the same straight line a' of the figure σ' We
thus two figures σ_1 and σ_2 lying in the same plane σ
so related that to the points A_1, B_1, C_1, and the
A_1B_1, A_1C_1, , B_1C_1, , of the one correspond the p
A_2, B_2, C_2, , and the lines A_2B_2, A_2C_2, , B_2C_2, , of the
Since any two corresponding straight lines of σ' and σ_1 int
in a point lying on the straight line $\sigma\sigma'$, and again an
corresponding straight lines of σ' and σ_2 intersect in a
lying on the same straight line $\sigma\sigma'$, it follows that
corresponding straight lines of σ, σ_1, and σ_2 meet in
and the same point, which is determined as the interse
of the straight line of σ' with the straight line $\sigma\sigma'$ T
to say, two corresponding straight lines of the figures σ
σ_2 always intersect on a fixed straight line, the trace of σ'
If moreover A_1 and A_2 are a pair of corresponding points
and σ_2, the rays S_1A_1, S_2A_2 have a point A' in common
therefore lie in the same plane consequently A_1A_2 and
intersect in a point O Thus we arrive at the property
every straight line such as A_1A_2, which connects a p

corresponding points of the figures σ_1 and σ_2, passes through a fixed point O, which is the intersection of S_1S_2 and σ From this we conclude that two figures σ_1 and σ_2 which are the projections of one and the same figure on one and the same plane, but from different centres of projection, possess all the properties of figures in perspective (Art 8) although they lie in the same plane To the points and the straight lines of the first correspond, each to each, the points and the straight lines of the second figure, two corresponding points always lie on a ray passing through a fixed point O, and two corresponding straight lines always intersect on a fixed straight line s Such figures are said to be *homological, or in homology*, O is termed the *centre of homology*, and s the *axis of homology** They may also be said to be *in plane perspective*, O being called the *centre of perspective*, and s the *axis of perspective*

19. Theorem *In the plane σ are given two figures σ_1 and σ_2 which are such that to the points A_1, B_1, C_1, and to the straight lines A_1B_1, A_1C_1, , B_1C_1, of the one correspond, each to each, the points A_2, B_2, C_2, and the straight lines A_2B_2, A_2C_2, , B_2C_2, of the other If the points of intersection of corresponding straight lines lie on a fixed straight line, then the straight lines which join corresponding points will all pass through a fixed point O*

Let A_1 and A_2, B_1 and B_2, C_1 and C_2 be three pairs of corresponding points, they form two triangles $A_1B_1C_1$ and $A_2B_2C_2$ whose corresponding sides B_1C_1 and B_2C_2, C_1A_1 and C_2A_2, A_1B_1 and A_2B_2 intersect in three collinear points By the theorem of Art 17 the rays A_1A_2, B_1B_2, C_1C_2 will therefore meet in the same point O, but two rays A_1A_2 and B_1B_2 suffice to determine this point, in whatever way then the third pair of points C_1, C_2 may be chosen, the ray C_1C_2 will always pass through O

The figures σ_1, σ_2 are therefore in homology, O being the centre, and s the axis, of homology

Corollary—It follows that if two figures lying either in the same or in different planes are in perspective, and if the plane of one of the figures be made to turn round the axis of perspective, then corresponding straight lines A_1A_2, B_1B_2, &c, will always be

* Poncelet, *Propriétés projectives*, Arts 297 seqq

concurrent, *i e* the two figures will remain always in perspecti The centre of perspective will of course change its position, it will seen further on (Art 22) that it describes a certain circle

20 THEOREM *If to the straight lines a, b, c, and to points ab, ac, , bc, , of a figure correspond severally straight lines a', b', c', and the points $a'b'$, $a'c'$, , $b'c'$, of another coplanar figure, so that the pairs of corresponding poi ab and $a'b'$, ac and $a'c'$, bc and $b'c'$, are collinear with fixed point O, then the corresponding straight lines a and b and b', c and c', will intersect in points which lie on a strai line*

Let a and a', b and b', c and c' be three pairs of cor sponding straight lines, since by hypothesis the straight li which join the corresponding vertices of the triangles abc, a' all meet in a point O, it follows (Art 16) that the correspoi ing sides a and a', b and b', c and c' intersect in three poi lying on a straight line But two points aa', bb', suffice determine this straight line, it remains therefore the sam instead of c and c' any other two corresponding rays considered Two corresponding straight lines therefore alw intersect on a fixed straight line, which we may call s, t the given figures are in homology, O being the centre, an the axis, of homology

21 Consider two homological figures σ_1 and σ_2 lying the plane σ, let O be their centre, s their axis of homolc Through the point O and outside the plane σ draw straight line, and on this take a point S_1, from which centre project the figure σ_1 upon a new plane σ' drawn in way through s In this manner we construct in the plane figure $A'B'C'$ which is in perspective with the given $\sigma_1 \equiv A_1B_1C_1$ If we consider two points A' and A_2 of figures σ' and σ_2, which are derived from one and the sa point A_1 of σ_1, as corresponding to each other, then to ev point or straight line of σ' corresponds a single point or strai line of σ_2, and *vice versa*, and every pair of correspond straight lines, such as $A'B'$ and A_2B_2, intersect on a fi straight line $\sigma\sigma'$ or s Consequently (Art 15) the figure and σ_2 are in perspective, and the rays $A'A_2$, $B'B_2$, pass through a fixed point S_2 Moreover every ray meets the straight line OS_1, since the points A', A_2 lie on

sides S_1A_1, OA_1 of the triangle OA_1S_1 The rays $A'A_2$, $B'B_2$, do not all lie in the same plane, because the points A_2, B_2, lie arbitrarily in the plane σ, the point S_2 therefore lies on the straight line OS_1

From this we conclude that two homological figures may be regarded, in an infinite number of ways, as the projections, from two distinct points, of one and the same figure, this figure lying in a plane passing through the axis of homology, and the two points being collinear with the centre of homology

22. Consider two figures in perspective, lying in the planes σ, σ' respectively (or two figures in plane perspective in the same plane σ), let O (Fig 4) be the centre and s the axis of perspective, and let j and i' be the vanishing lines of the two figures If J and I' are points lying on these vanishing lines, the points J' and I which correspond to each of them respectively in the other figure will be at infinity on the [illegible]pectively Further, the two corresponding [illegible], $I'J'$ must meet in some point on s, there are an infinite number of parallelograms having one [illegible], the opposite one on s, and the other two vertices on j and i' respectively

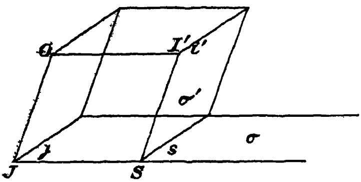


Fig 4

Now, supposing the two figures to keep their positions in their planes unaltered, let the plane σ' be made to turn round $\sigma\sigma'$ or s Every pair of corresponding straight lines must always meet on s, consequently the two figures will always remain in perspective (Arts 15, 19), and the point O will describe some curve in space

In order to determine this curve, consider any one of the above mentioned parallelograms $OJSI'$ It remains always a parallelogram, and the length of $I'S$ is invariable, therefore also OJ is of constant length The locus of the centre of perspective O is therefore a circle whose centre lies on the vanishing line j and whose plane is perpendicular to this line and therefore to the axis of perspective s *

* MÖBIUS, *Barycentrische Calcul* (Leipzig, 1827), § 230 (note p 326)

23 (1) *Given the centre O and the axis s of homology, and tu corresponding points A and A′ (collinear with O), to construct ti figure homological with a given figure*

Take a second point B of the given figure (Fig 5) To obtain th corresponding point B', we notice that the ray BB' must pass throug O and that the straight lines AB, $A'B'$ which correspond to or another must intersect on s, thus B' will be the point where O meets the straight line joining A' to the intersection of AB with s In the same way we can construct any number of pairs of correspon ing points, in order to draw the straight line r' which corresponds to a given straight line r, we have only to find the point B' which corresponds to a point B lying on the line r, and to join the points B' and rs

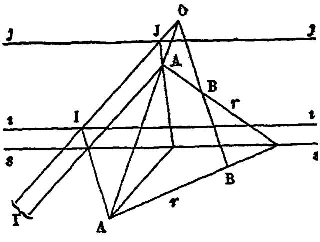


Fig 5

In order to find the point I' (the vanishing point) which corresponds to the infinitely distant point I on a given straight line (a ray OI, for example, drawn frc O), we repeat the construction just given for the point B', i e we jc another point A of the first figure to the point at infinity I on ((that is, we draw AI parallel to OI), and then join A' to the poi where AI meets s, and produce the joining line to cut OI in Then I' is the required point

All points analogous to I' (i e those which correspond to the poi at infinity in the given figure) fall on a straight line i', parallel to s, i' is the vanishing line of the second figure If, in the preceding construction, we interchange the points A and A'†, we shall obtain a point J (a vanishing point) lying on the vanishing line j of the first figure

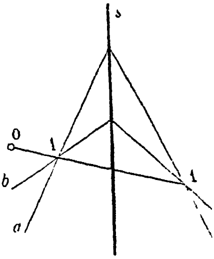


Fig 6

(2) Suppose that instead of two corresponding points A A' there are given (Fig 6) *two corresponding straight lines* a, a' These will of course intersect on s, and every ray passing through O will cut them in two correspondi

* This construction shows that if B lies upon s, then B will coincide with i e that every point of s is its own correspondent

† Otherwise Draw through A any straight line JA, then through A the intersection of $J'A'$ with s draw a straight line JA, and through O draw parallel to $A'J'$ Then the intersection of OJ and JA is the vanishing point and a straight line j drawn through J parallel to s is the vanishing line of first figure

points A, A' In order to obtain the straight line b' which corresponds to any straight line b in the first figure, we have only to join the point bs to the point of intersection of a' with the ray passing through O and ab *

(3) The data of the problem may also be *the centre O, the axis s, and the vanishing line j of the first figure* (Fig 7 In this case, if a straight line a of the first figure cuts j in J and s in P, the point J' corresponding to J will be collinear with J and O and at an infinite distance from O And as the straight line a' corresponding to a must pass both through J' and through P it is the parallel drawn through P to OJ

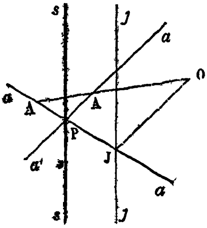


Fig 7

To find the point A' corresponding to given point A, we must draw the straight line a' which corresponds to a straight line a drawn arbitrarily through A, the intersection of a' with OA is the required point A'

(4) Assuming a knowledge of the constructions just given, let again O be the centre, s the axis, of homology, and j the vanishing line of the first figure.

In the first figure let a circle C be given (Figs 8, 9, 10), to the circle will correspond in the second figure a curve C' which we can construct by determining, according to the method above, the points and straight lines which correspond to the points and tangents of C

Two corresponding points will always be collinear with O, and two corresponding chords (*i e* straight lines MN, $M'N'$, where M and M', N and N', are two pairs of corresponding points) will always intersect on s, as a particular case two corresponding tangents m and m' (*i e* tangents at corresponding points M and M') will meet in a point lying on s

It follows clearly from this that the curve C' possesses, in common with the circle, the two following properties

(1) Every straight line in its plane either cuts it in two points, or is a tangent to it, or has no point in common with it

(2) Through any point in the plane can be drawn either two tangents to the curve, or only one (if the point is on the curve) or none

Since two homological figures can be considered as arising from the superposition of two figures in perspective lying in different planes (Art 22) *the curve C' is simply the plane section of an oblique cone on a circular base i e the cone which is formed by the straight lines which run from any point in space to all points of a circle*

* It follows from this that if a passes through O, then a' will coincide with a i e every straight line passing through O corresponds to itself

For this reason the curve C' is called a *conic section* or simpl *conic*, thus *the curve which is homological with a circle is a conic.*

The points on the straight line j correspond to the points infinity in the second figure Now the circle C may cut j in t

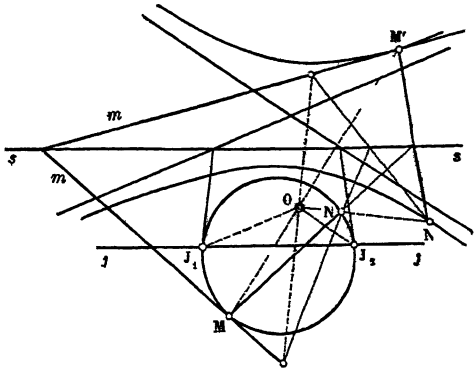


Fig 8

points J_1, J_2 (Fig 8), or it may touch j in a single point J (Fig or it may have no point in common with j (Fig 10)

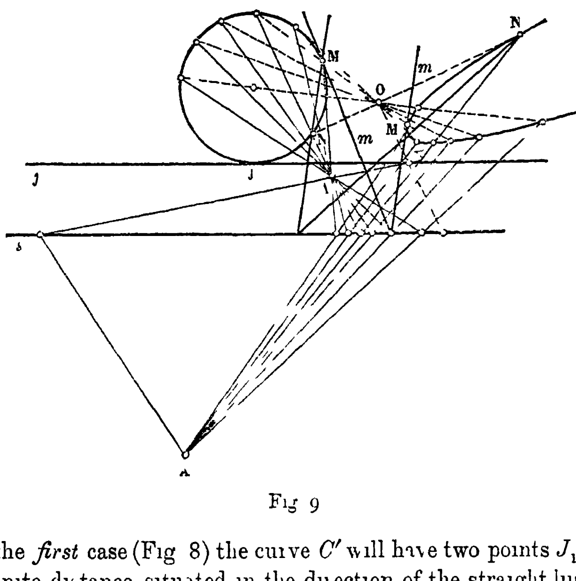


Fig 9

In the *first* case (Fig 8) the curve C' will have two points J_1', J an infinite distance, situated in the direction of the straight lines C OJ_2 To the two straight lines which touch the circle in J_1 and will correspond two straight lines (parallel respectively to OJ_1

OJ_2) which must be considered as tangents to the curve C' at its points at infinity J_1', J_2' These two tangents, whose points of contact lie at infinity, are called *asymptotes* of the curve C', the curve itself is called a *hyperbola*

In the *second* case (Fig 9) the curve C' has a single point J' at infinity, this must be regarded as the point of contact of the straight line at infinity j', which is the tangent to C' corresponding to the

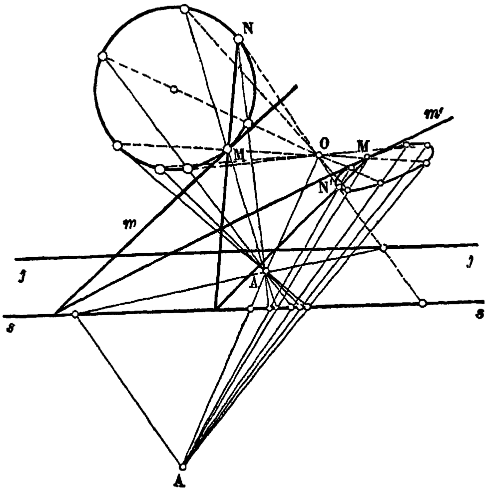


Fig 10

tangent j at the point J of the circle This curve C is called a *parabola*

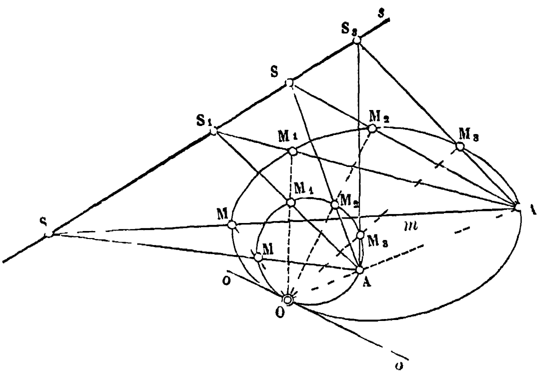


Fig 11

In the *third* case (Fig 10) the curve has no point at infinity, it is called an *ellipse*

In the same way it may be shown that if in the first figure a C is given, the corresponding curve C' in the second figure will conic also

(5) The centre of homology is a point which corresponds to and every ray which passes through it corresponds to itself. If a curve C pass through O, the corresponding curve C' will also through O and the two curves will have a common tangent a point Fig 11 shows the case where one of the curves is taken a circle, and the axis of homology s and the point A correspond the point A' of the circle are supposed to be given.

Similarly, every point on the axis of homology correspo itself If then a curve belonging to the first figure touch a certain point, the corresponding curve in the second figure will s at the same point In Fig 12 is shown a circle which is transformed homologically by means of its tangents, moreove

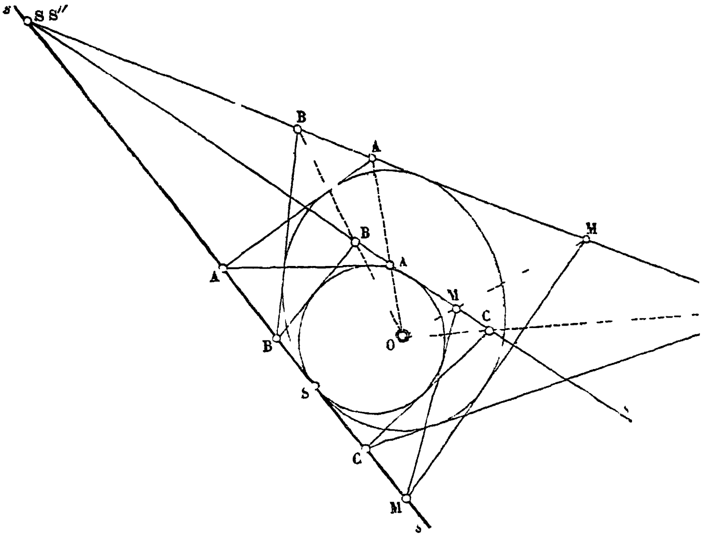


Fig 12

supposed that the axis of homology touches the circle, that the of homology is any given point, and that the straight line a second figure is given which corresponds to the tangent a' circle

(6) Two particular cases may be noticed

(1) The axis of homology s may lie altogether at infinity, th corresponding straight lines are always parallel or, what amo the same thing, two corresponding angles are always equal

case the two figures are said to be *similar and similarly placed*, or *homothetic* *, and the point O is called the *centre of similitude*

Let M_1, M_1' and M_2, M_2' be two pairs of corresponding points of two homothetic figures, so that M_1M_1', M_2M_2' meet in O, while M_1M_2, $M_1'M_2'$ are parallel By similar triangles

$$OM_1 \quad OM_1' = OM_2 \quad OM_2' = M_1M_2 \quad M_1'M_2',$$

that *the ratio OM OM' is constant for all pairs of corresponding* *points M and M'* This constant ratio is called the *ratio of similitude* of the two figures

The tangents at two corresponding points M, M' must meet on the axis of homology s, *i e* they are parallel to one another If then the tangent at M pass through O, it must coincide with the tangent at M' It follows that if the two figures are such that common tangents can be drawn to them, *every common tangent passes through a centre of similitude*

Take two points C, C' collinear with O and such that

$$\frac{OC}{OC'} = \frac{OM}{OM'} = \text{ratio of similitude}$$

Then if CM, $C'M'$ be joined, they will evidently be parallel, and CM CM' = ratio of similitude Therefore if M lie on a circle, centre C and radius ρ, M' will lie on another circle whose centre is C' and whose radius ρ' is such that ρ ρ' = ratio of similitude In two homothetic figures then to a circle always corresponds a circle [Further, if CC' be again divided at O', so that

and clear

$$O'C \quad O'C' = OC \quad OC' = \rho \quad \rho' = \text{ratio of similitude},$$

it is clear that O' will be a second centre of similitude for the two circles It can be proved in a similar manner that any two central conics (see Chap XXI) which are homothetic, and for which a point O is the centre of similitude, have a second centre of similitude O', and that O, O' are collinear with the centres C, C' of the two conics, and divide the segment CC' internally and externally in the ratio of similitude If the conics have real common tangents, O and O' will be the points of intersection of these taken in pairs—the two external tangents together, and the two internal tangents together

(2) The point O, on the other hand, may lie at an infinite distance, then the straight lines which join pairs of corresponding points are parallel to a fixed direction In this case the figures have been termed *homological by affinity* †, the straight line s being termed the *axis of*

* Homothetic figures may be regarded as sections of a pyramid or a cone made by parallel planes s, the line of intersection of the two planes, lies at an infinite distance This is the case in Art 8 if σ and σ' are parallel planes

† EULER, *Introductio* ii cap 18, MOBIUS, *Baryc Calcul*, § 144 et seqq

affinity * To a point at infinity corresponds in this case a poi
infinity, and the straight line at infinity corresponds to itsel
follows from this that to an ellipse corresponds an ellipse, to a h
bola a hyperbola, to a parabola a parabola, to a parallelogr
parallelogram

* If two figures are so related, they may be regarded as plane section
prism or of a cylinder This is the case in Art. 8 if the centre S of projec
infinitely distant The projection is then called *parallel projection.*
particular case where the parallels SA, SB, SC, are perpendicular to the
of projection it is called *orthogonal projection.*

blem
Construct an ellipse, parabola, hyp
homological to given circles

CHAPTER IV.

HOMOLOGICAL FIGURES IN SPACE

24. SUPPOSE a figure to be given which is made up of point planes, and straight lines lying in any manner in space, the *relie perspective** of this is constructed in the following manner A poi O in space is taken as *centre of perspective* or *homology*, a *plane homology* π is taken, every point of which is to be its own image and in addition to these is taken a point A' which is to be the imag of a point A of the given figure, so that AA' passes through O L now B be any other point, in order to obtain its image B', the plai OAB is drawn, and we then proceed in this plane as if we had construct two homological figures, taking O as the centre and th intersection of the planes OAB and π as the axis of homology, and A, A as two corresponding points The point B' will be the intersection OB with the straight line passing through A' and the point where th straight line AB cuts the plane π (Art 23, Fig 4) Let C be a thi point, its image C' will be the point of intersection of OC wi $A'D$ or with $B'E$ (in π), where D and E are the points in whi the plane π is met by AC, BC respectively

This method will yield, for every point of the given figure, th corresponding point of the image, and two corresponding points w always lie on a straight line passing through O Every plane passing through O cuts the two solid figures (the given one and i image) in two homological figures, for which O is the centre, and th straight line $\sigma\pi$ the axis, of homology It follows from this that every straight line of the given figure corresponds a straight line the image, and that two corresponding straight lines lie always in plane passing through O and meet each other in a point lying on th plane π

Further to every plane a, belonging to the given figure, and n passing through O, will correspond a plane a' in the image For to th straight lines a, b, c, of the plane a correspond severally the straig

* This problem may present itself in the construction of bas reliefs and theatre decorations (PONCELET, *Prop proj* 584, POUDRA, *Perspective reli* Paris, 1860)

lines $a', b', c',$, and to the points $ab, ac,$, $bc,$ the points $a'b',$, $b'c',$ In other words, the straight lines $a', b', c',$ are s that they intersect in pairs, but do not all meet in the same poi they lie therefore in the same plane a' * Two corresponding pla a, a' intersect on the plane π, for all the points and all the strai lines of this last plane correspond to themselves, and therefore straight line $a'\pi$ coincides with the straight line $a\pi$

The two planes a, a' evidently contain two figures in perspec (like the planes σ, σ' of Arts 12 and 14).

25 In every plane σ passing through O lies a vanishing line which is the image of the point at infinity in the same plane vanishing lines of the planes σ_1, σ_2 have a common point, which is image of the point at infinity on the line $\sigma_1\sigma_2$. The vanishing l of all the planes σ are therefore such as to cut each other in pa and as they do not pass all through the same point (since the pl through O do not pass all through the same straight line), they m lie in one and the same plane ϕ'

This plane ϕ', which may be called the *vanishing plane*, is par to the plane π, since all the vanishing lines of the planes σ parallel to the same plane π The vanishing plane ϕ' is thus locus of the straight lines which correspond to the straight line infinity in all the planes of space, and is consequently also the l of the points which correspond to the points at infinity in all straight lines of space for the line at infinity in any plane a is same thing as the line at infinity in the plane through O paralle a, so also the point at infinity on any straight line a coincides the point at infinity on the straight line drawn through O par to a

26 The infinitely distant points of all space are then such their images are the points of one and the same plane ϕ' (the vanis plane) It is therefore natural to consider all the infinitely di points in space as lying in one and the same plane ϕ (the plan infinity) of which the plane ϕ' is the image †

The idea of the plane at infinity being granted the point at infi on any straight line a is simply the point $a\phi$, and the straight li infinity in any plane a is the straight line $a\phi$ Two straight lines parallel if they intersect in a point of the plane ϕ, two planes parallel if their line of intersection lies in the plane ϕ, &c

* Since c cuts both a' and b' without passing through the point $a'b'$, ther c' has two points in common with the plane $a'b'$ and consequently lies entire the plane $a'b'$ And similarly for the other straight line

† PONCELET, *Prop proj* 580

CHAPTER V

GEOMETRIC FORMS

27 A *range* or *row* of points is a figure $A, B, C, \ldots$ compo of points lying on a straight line (which is called the *base* the range), such is, for example, the figure resulting from t operations of Art. 5 or Art 7

An *axial pencil* is a figure $\alpha, \beta, \gamma, \ldots$ composed of planes passing through the same straight line (the *axis* of the penc such is the figure resulting from the operations of Art 4 Art 6

A *flat pencil* is a figure $a, b, c, \ldots$ composed of straight li lying all in the same plane and radiating from a given po (the *centre* or *vertex* of the pencil), such would be the fig obtained by applying the operation of Art 2 to a range, that of Art 3 to an axial pencil

A *sheaf* (*sheaf of planes*, *sheaf of lines*) is a figure made up planes or straight lines, all of which pass through a giv point (the *centre* of the sheaf), like that which results fr the operation of Art 2

A *plane figure* (*plane of points*, *plane of lines*) is a figure wh consists of points or straight lines all of which lie in the sa plane, such is the figure resulting from the operation Art 3

28 The first three figures can be derived one from the ot by a projection or a section*

From a range $A, B, C, \ldots$ is derived an axial pen $s(A, B, C, \ldots)$ by projecting the range from an axis s (Art and a flat pencil $O(A, B, C, \ldots)$ by projecting it from a cen

* The series of planes $sA, sB, sC \ldots$, of rays $OA, OB, OC, \ldots$, of points β γ $\ldots$, and of straight lines $\sigma\alpha, \sigma\beta, \sigma\gamma, \ldots$ will be denoted by $s(A, B, C,$ $O(A, B, C, \ldots)$, $s(\alpha, \beta, \gamma, \ldots)$, and $\sigma(\alpha, \beta, \gamma, \ldots)$ respectively To denote ries of points A B $C, \ldots$ the symbols $A, B, C, \ldots$ and $ABC \ldots$ will be indifferently

O (Art 2) From an axial pencil $\alpha, \beta, \gamma,$ is derived a r $s\ (\alpha, \beta, \gamma, \quad)$ by cutting the pencil by a transversal l (Art 5), and a flat pencil $\sigma\ (\alpha, \beta, \gamma, \quad)$ by cutting it transversal plane σ (Art 3) From a flat pencil $a, b, c,$ derived a range $\sigma\ (a, b, c, \quad)$ by cutting it by a transv plane σ (Art 3), and an axial pencil $O\ (a, b, c, \quad)$ by jecting it from a centre O (Art. 2)

29 In a similar manner the last two figures of Art. 2 be derived one from the other by help of one of the opera of Art 2 or Art 3, in fact, if we project from a centre plane of points or lines we obtain a sheaf of ~~lines or pla~~ and reciprocally, if we cut a sheaf of lines or planes transversal plane we obtain a plane of points or lines plane figures in perspective (Art. 12) are two sections o same sheaf

30 The *elements* or *constituents* of the range are the po those of the axial pencil, the planes, those of the flat p the straight lines or rays

In the plane figure either the points or the straigh may be regarded as the elements If the points are cons as the elements, the straight lines of the figure are so ranges, if, on the other hand, the straight lines or ray considered as the elements, the points of the figure ar centres of so many flat pencils

The plane of points (*i e* the plane figure in which th ments are points) contains therefore an infinite numb ranges*, and the plane of lines (*i e* the plane figure in w the elements are lines †) contains an infinite number o pencils

In the sheaf either the planes, or the straight lines or may be regarded as the elements If we take the plan elements, the rays of the sheaf are the axes of so axial pencils, if, on the other hand the rays are cons as the elements, the planes of the sheaf are so many pencils

The sheaf contains therefore an infinite number of

* One of these ranges has all its points at an infinite distance, each others has only one point at infinity

† The straight line at infinity belongs to an infinite number of flat penci of which has its centre at infinity, and consequently all its rays parallel

pencils or an infinite number of flat pencils, according as planes or its straight lines are regarded as its elements.

31 Space may also be considered as a geometrical figı whose elements are either points or planes

Taking the points as elements, the straight lines of sp are so many ranges, and the planes of space so many plane points If, on the other hand, the planes are considered elements, the straight lines of space are the axes of so ma axial pencils, and points of space are the centres of so ma sheaves of planes

Space contains therefore an infinite number of planes points* or an infinite number of sheaves of planes †, accord as we take the point or the plane as the element in ordeı construct it.

32. The first three figures, viz the range, the axial pen and the flat pencil, which possess the property that each be derived from the other by help of one of the operations Arts 2, 3, , are included together under one name, and termed the *one-dimensional geometric prime-forms*

The fourth and fifth figures, viz the sheaf of planes or lı and the plane of points or lines, which may in like manner derived one from the other by means of one of the operatı of Arts 2, 3, , and which moreover possess the property including in themselves an infinite number of one-dimensio prime-forms, are likewise classed together under one title, the *two-dimensional geometric prime-forms*

Lastly, space, which includes in itself an infinite numbeı two-dimensional prime-forms, is considered as constituting *three-dimensional geometric prime-form*

There are accordingly six geometric prime-forms , three one dimension, two of two dimensions, and one of th dimensions ‡

Note—With reference to the use of the word *dimension* in preceding Article, it is clear, from what has been said in Art that we are justified in considering the range, the flat pencil, the axial pencil, as of the same dimensions, since to every point

* One of them lies entirely at infinity

† Among these, there are an infinite number which have their centre a infinite distance, and whose rays are consequently parallel

‡ v STAUDT, *Geometrie der Lage* (Nurnberg 1847) Arts 26 28

the first corresponds one ray in the second and one plane i third The number of elements in each of these forms is in but it is the same in all three

Similarly we conclude from Art 29 that we are justified ir sidering the plane figure as of the same dimensions with the she

But the plane of points (lines) contains (Art. 30) an infinite nı of ranges (flat pencils), and each of these ranges (flat pencils) contains an infinite number of points (rays) Thus the plane contains a number of points (lines) which is an infinity of the s order compared with the infinity of points in a range, or of ray flat pencil, and must therefore be considered as of two dimensi the range and flat pencil are taken to be of one dimension

So too the sheaf of planes (or lines) contains (Art. 30) an i number of axial pencils (or of flat pencils), and each of these contains an infinite number of planes (or of rays) Therefor the sheaf of planes or lines must be of double the dimensions axial pencil or the flat pencil

Again, space, considered as made up of points, contains an i number of planes of points, and considered as made up of pla contains an infinite number of sheaves of planes Space thus cc an infinite number of forms of two dimensions, which latter, contain each an infinite number of forms of one dimension must accordingly be regarded as of three dimensions

We may put the matter thus

Forms of one dimension are those which contain a simple i (∞) of elements,

Forms of two dimensions are those which contain a double i (∞^2) of elements,

Forms of three dimensions are those which contain a triple i (∞^3) of elements

CHAPTER VI

THE PRINCIPLE OF DUALITY *

33 GEOMETRY (speaking generally) studies the generation and the properties of figures lying (1) in space of three dimensions, (2) in a plane, (3) in a sheaf In each case, any figure considered is simply an assemblage of elements, or, what amounts to the same thing, it is the aggregate of the elements with which a moving or variable element coincides in its successive positions The moving element which generates the figures may be, in the first case, the point or the plane, in the second case the point or the straight line, in the third case the plane or the straight line There are therefore always two *correlative* or *reciprocal* methods by which figures may be generated and their properties deduced, and it is in this that geometric *Duality* consists By this duality is meant the co-existence of figures (and consequently of their properties also) in pairs, two such co-existing (*correlative* or *reciprocal*) figures having the same genesis and only differing from one another in the nature of the generating element

In the Geometry of space the range and the axial pencil, the plane of points and the sheaf of planes, the plane of lines and the sheaf of lines, are correlative forms The flat pencil is a form which is correlative to itself

In the Geometry of the plane the range and the flat pencil are correlative forms

In the Geometry of the sheaf the axial pencil and the flat pencil are correlative forms

The Geometry of the plane and the Geometry of the sheaf, considered in three-dimensional space, are correlative to each other

34 The following are examples of correlative propositions

* S[illegible] *Geo[illegible] der La[illegible]e* Art 66

in the Geometry of space Two correlative propositions a deduced one from the other by interchanging the elemen *point* and *plane*

3 1 Two points A, B determine a straight line (viz the straight line AB which passes through the given points) which contains an infinite number of other points

2 A straight line a and a point B (not lying on the line) determine a plane, viz the plane aB which connects the line with the point

5 3 Three points A, B, C which are not collinear determine a plane, viz the plane ABC which passes through the three points

4 4 Two straight lines which cut one another lie in the same plane

5 Given four points A, B, C, D, if the straight lines AB, CD meet, the four points will lie in a plane, and consequently the straight lines BC and AD, CA and BD will also meet two and two

6 Given any number of straight lines, if each meets all the others, while the lines do not all pass through a point, then they must lie all in the same plane (and constitute a plane of lines)*

1 Two planes α, β determine straight line (viz. the straight li $\alpha\beta$, the intersection of the give planes), through which pass infinite number of other planes.

2 A straight line a and a pla β (not passing through the lin determine a point, viz the poi $a\beta$ where the line cuts the plan

3 Three planes α, β, γ whi do not pass through the same li determine a point, viz the poi $\alpha\beta\gamma$ where the three planes me each other

4 Two straight lines which in the same plane int point

5 Given four planes α if the straight lines $\alpha\beta$, , the four planes will me a point, and consequently straight lines $\beta\gamma$ and $\alpha\delta$, $\gamma\alpha$ a $\beta\delta$, will also meet two and two

6 Given any number of straig lines, if each meets all the othe while the lines do not all lie the same plane, then they mu pass all through the same po (and constitute a sheaf of lines

7 The following problem admits of two correlative solutio Given a plane α and a point A in it, to draw through A a straig line lying in the plane α which shall cut a given straight line r whi does not lie in α and does not pass through A

* See note to Art 20

† For let a, b, c, be the straight lines, as ab, ac, bc are thr e planes disti from each other, the common point must be the intersection of the straight li a, b, c,

Join A to the point ra.

Construct the line of inte section of the plane a with tl plane rA

8 *Problem* Through a given point A to draw a straight line to cut each of two given straight lines b and c (which do not lie in the same plane and do not pass through A)

Solution Construct the line of intersection of the planes Ab, Ac.

8 *Problem* In a given plai a, to draw a straight line to c each of two given straight lines and c (which do not meet and not lie in the plane a)

Solution Join the point ab the point ac

35 In the Geometry of Space, the figure correlative to a triang (system of three points) is a trihedral angle (system of three planes the vertex, the faces, and the edges of the latter are correlative the plane, the vertices, and the sides respectively of the triangl thus the theorem correlative to that of Arts 15 and 17 will be t following

If two trihedral angles $\alpha'\beta'\gamma'$, $\alpha''\beta''\gamma''$ are such that the edges β' and $\beta''\gamma''$, $\gamma'\alpha'$ and $\gamma''\alpha''$, $\alpha'\beta'$ and $\alpha''\beta''$ lie in three planes α_0, β_0, which pass through the same straight line, then the straight lir $\alpha'\alpha''$, $\beta'\beta''$, $\gamma'\gamma''$ will lie in the same plane

The proof is the same as that of Arts 15 and 17, if the elemer point and plane are interchanged If, for example the two trihedi angles have different vertices S', S'' (Art 15), then the points whe the pairs of edges intersect are the vertices of a triangle whose sid are $\alpha'\alpha''$, $\beta'\beta''$, $\gamma'\gamma''$, these latter straight lines lie therefore in t same plane (that of the triangle)

So also the proof for the case where the two trihedral angles ha the same vertex S will be correlative to that for the analogous case two triangles $A'B'C'$ and $A''B''C''$ which lie in the same plane (A 17) The theorem may also be established by projecting from a poi S the figure corresponding to the theorem of Art 16

The proof of the theorem correlative to that of Arts 14 and 16 left as an exercise for the student It may be enunciated as follow

If two trihedral angles $\alpha'\beta'\gamma'$, $\alpha''\beta''\gamma''$ are such that the straight li $\alpha'\alpha''$, $\beta'\beta''$, $\gamma'\gamma''$ lie in the same plane, then the pairs of edges $\beta'\gamma'$ a $\beta''\gamma''$, $\gamma'\alpha'$ and $\gamma''\alpha''$, $\alpha'\beta'$ and $\alpha''\beta''$ determine three planes which p all through the same straight line

36 In the Geometry of the plane, two correlative prop sitions are deduced one from the other by interchanging t words *point* and *line*, as in the following examples

1 Two points A, B determine a straight line, viz the line AB

2 Four points A, B, C, D (Fig 13), no three of which are collinear, form a figure called a *complete quadrangle** The four

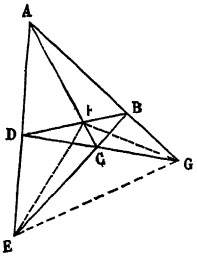


Fig 13

points are called the *vertices*, and the six straight lines joining them in pairs are called the *sides* of the quadrangle

Two sides which do not meet in a vertex are termed *opposite*, there are accordingly three pairs of opposite sides, BC and AD, CA and BD, AB and CD The

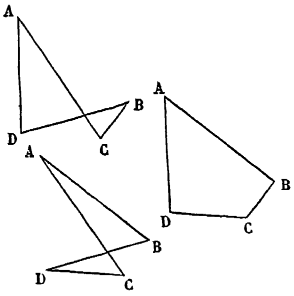


Fig 15

points E, F, G in which the opposite sides intersect in pairs are

1 Two straight lines a, b de-
termine a point, viz. the p
ab

2 Four straight lines a, b,
(Fig. 14), no three of which
concurrent, form a figure call
*complete quadrilateral**. The

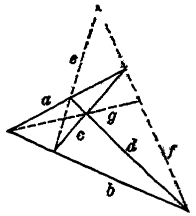


Fig 14.

straight lines are called the
of the quadrilateral, and the
points in which the sides cut
another two and two are c
the *vertices*

Two vertices which do no
on the same side are te
opposite, there are accord
three pairs of opposite vertic
and ad, ca and bd, ab an

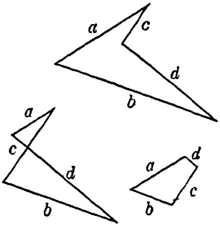


Fig 16

The straight lines e, f, g
join pairs of opposite vertice

* The complete quadrangle has also been called a *tetrastigm*, and the co
quadrilateral a *tetragram* TOWNSEND, *Modern Geometry*, ch vii

termed the *diagonal points*, and the triangle EFG is termed the *diagonal triangle* of the complete quadrangle. The complete quadrangle includes three simple quadrangles, viz. $ACBD$, $ABCD$, and $ABDC$ (Fig. 15)

called the *diagonals*, and the triangle efg is termed the *diagonal triangle* of the complete quadrilateral. The complete quadrilateral includes three simple quadrilaterals, viz. $acbd$, $adcb$, and $acbd$ (Fig. 16)

3. And so, in general

A *complete polygon* (complete *n-gon*, or *n-point**) is a system of n points or *vertices*, with the $\frac{n(n-1)}{2}$ straight lines or *sides* which join them two and two

A *complete multilateral* (or *n-side*†) is a system of n straight lines or *sides*, with the $\frac{n(n-1)}{2}$ points or *vertices* in which they intersect one another two and two

4. The theorems of Arts 16 and 17 are correlative each to the other

5. **Theorem** If two complete quadrangles $ABCD$, $A'B'C'D'$ are such that five pairs of sides AB and $A'B'$, BC and $B'C'$, CA and $C'A'$, AD and $A'D'$, BD and $B'D'$ cut one another in five points lying on a straight line s, then the remaining pair CD and $C'D'$ will also intersect one another on s (Fig 17)

Theorem If two complete quadrilaterals $abcd$, $a'b'c'd'$ are such that five pairs of vertices ab and $a'b'$, bc and $b'c'$, ca and $c'a'$, ad and $a'd'$, bd and $b'd'$ lie upon five straight lines which meet in a point S, then the remaining pair cd and $c'd'$ will also lie on a straight line through S (Fig 18)

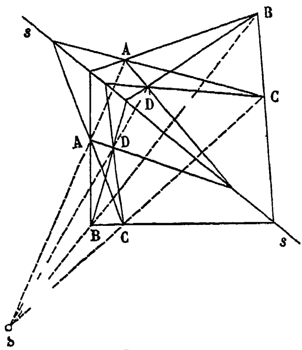


Fig 17

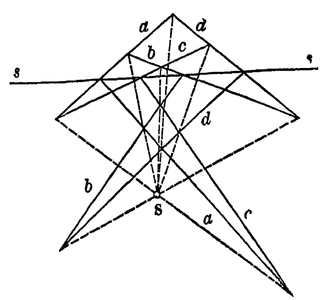


Fig 18

Since the triangles ABC, $A'B'C'$ are by hypothesis in

Since the triangles (trilaterals) abc, $a'b'c'$ are

* Or *polystigm* TOWNSEND *loc cit*

† Or *polygram*

perspective (Arts 17, 18), the straight lines AA', BB', CC' will meet in one point S So too the triangles ABD, $A'B'D'$ are in perspective, therefore DD' also will pass through S, the point common to AA' and BB' It follows that the triangles BCD, $B'C'D'$ are also in perspective therefore CD and $C'D'$ meet in a point on the straight line s, which is determined by the point of intersection of BC and $B'C'$ and by that of BD and $B'D'$*

hypothesis in perspectiv 18), the points aa', l will lie on one straight So too the triangles abd, a in perspective, therefore t dd' lies on the straight which passes through th aa', bb' It follows t triangles (trilaterals) bc are also in perspective, t cd and $c'd'$ lie on a strai through the point S, w determined by the straig (bc) $(b'c')$ and (bd) $(b'd')$*

37 In the Geometry of space the following are correlative

A complete *n-gon* (in a plane)

A complete *n-flat* (in s *i e.* a figure made up of (or *faces*) which all pass the same point (or *verte* ther with the $\frac{n(n-1)}{2}$ which these planes inte and two

A complete *multilateral of n sides*, or *n side* (in a plane)

A complete *n-edge* (in *i e* a figure made up of n lines radiating from a point (or *vertex*), togetl the $\frac{n(n-1)}{2}$ planes (c which pass through these lines taken in pairs

Thus the following theorems are correlative, in the Ge space, to the two theorems above (Art 36, No 5), whi are themselves correlative to each other in the Geometr plane

If two complete four-flats in a sheaf (be their vertices coincident or not) $\alpha\beta\gamma\delta$, $\alpha'\beta'\gamma'\delta'$ are such that five pairs of corresponding

If two complete four e sheaf (be their vertices c or not) $abcd$, $a'b'c'd'$ are five pairs of correspond

* These two theorems hold good equally when the two quadrangles laterals lie in different planes, in fact, the proofs are the same as the a for word

edges lie in five planes which pass all through the same straight line s, then the sixth pair of corresponding edges will lie also in a plane passing through s

cut one another in five straight lines which lie all in one plane σ, then the line of intersection of the sixth pair of corresponding faces will lie also in the plane σ

The proofs of these theorems are left as an exercise to the student They only differ from those of the theorems No 5, Art 36 in the substitution for each other of the elements point and plane, and just as theorems 5, Art 36 follow from those of Arts 15 and 16, so the theorems enunciated above follow from those of Art 35 When the two four-flats have the same vertex O, the theorem on the left-hand side may also be established by projecting from the point O (Art 2) the figure corresponding to the right-hand theorem of No 5, Art 36 And in this case we may by the same method deduce the theorem on the right-hand side above from that on the left-hand of No 5, Art 36

38 In the Geometry of the sheaf, two correlative theorems are derived one from the other by interchanging the elements plane and straight line Just as the Geometry of the sheaf is correlative to that of the plane, with regard to three-dimensional space, so one of the Geometries is derived from the other by the interchange of the elements point and plane The Geometry of the sheaf may also be derived from that of the plane by the operation of projection from a centre (Art 2)

From the Geometry of the sheaf may be derived that of spherical figures, by cutting the sheaf by a sphere passing through the centre of the sheaf

one can be obtained from
ter by a finite no. of proje
d sections.

Two geometric figures are pe
one can be obtained from the
a single projection, or by a pro
and secti

CHAPTER VII

PROJECTIVE GEOMETRIC FORMS.

39 By means of projection from a centre we obtain f a range a flat pencil, from a flat pencil an axial pencil, fro plane of points or lines a sheaf of lines or planes. C versely, by the operation of section by a transversal pl we obtain from a flat pencil a range, from an axial penc flat pencil, from a sheaf a plane figure The two operati projection from a point and section by a transversal pl may accordingly be regarded as *complementary* to each otl and we may say that if one geometric form has been deri from another by means of one of these operations, we can versely, by means of the complementary operation, derive second form from the first And similarly for the operati projection from an axis and section by a transversal line

Suppose now that by means of a series of operations, eac which is either a projection or a section, a form f_2 has l derived from a given form f_1, then another form f_3 from f_2 so on, until by $n-1$ such operations the form f_n has l arrived at Conversely, we may return from f_n to f_1 by m of another series of $n-1$ operations which are complemen respectively to the last, last but one, last but two, &c of operations by which we have passed from f_1 to f_n The se of operations which leads from f_1 to f_n, and the series wl leads from f_n to f_1, may be called complementary, and operations of the one series are complementary respectivel those of the other, taken in the reverse order

In the above the geometric forms are supposed to li space (Art 31) If we confine ourselves to plane Geometry complementary operations reduce to *projection from a centre*

section by a transversal line In the Geometry of the sheaf, section by a plane and projection from an axis are complementary operations

40 Two geometric prime forms of the same dimensions are said to be *projectively related*, or simply *projective*, when one can be derived from the other by any finite number of projections and sections (Arts 2, 3, 7)

For example, let a range u be given, project it from a centre O, thus obtaining a flat pencil, project this flat pencil from another centre O', by which means an axial pencil with OO' as axis is produced, cut this axial pencil by a straight line u_2, thus obtaining a range of points lying on u_2, project this range from an axis, and cut the resulting axial pencil by a plane, by which means a flat pencil is produced, and so on, then any two of the one-dimensional geometric forms which have been obtained in this manner are projective according to definition.

When we say that a form $A, B, C, D,$ is projective with another form $A', B', C', D',$ we mean that, by help of the same series of operations, each of which is either a projection or a section, A' is derived from A, B' from B, C' from C, &c The elements A and A', B and B', C and C', are termed *corresponding elements* *

For example, a plane figure is said to be projective with another plane figure, when from the points $A, B, C,$ and from the straight lines $AB, AC, BC,$ of the one are derived the points $A', B', C',$ and the straight lines $A'B', A'C',$ $B'C',$ of the other, by means of a finite number of projections and sections

In two projective plane figures, to a range in the one corresponds in the other a range which is projective with the first range, and to a flat pencil in the one figure corresponds in the other a flat pencil which is projective with the first pencil

41 From what has been said above it is easy to see that two geometric forms which are each projective with

* Two projective forms are termed *homographic* when the elements of which they are constituted are of the same kind, *i e* when the elements of *both* are points or lines, or planes It will be seen later on (Art 67) that this definition of homography is equivalent to that given by CHASLES (*Géométrie supérieure* Art 99)

a third are projective with one another For if we first through the operations which lead from the first form to the third, and then go through those which lead from the third the second, we shall have passed from the first form to the second

42 *Geometric forms in perspective*

The following forms are said to be *in perspective*

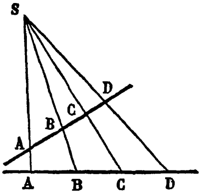


Fig 19

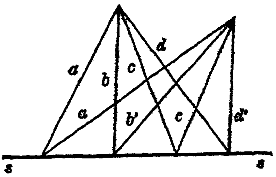


Fig 20.

Two ranges (Fig 19), if they are sections of the same flat pencil (Art 12)

Two flat pencils (Fig 20), if they project, from different centres, one and the same range, or if they are sections of the same axial pencil.

[*Note* —If we project a range $u \equiv ABC$ from two different centres O and O' not lying in the same plane with it, we obtain two pencils in perspective These pencils, again, may be regarded as sections of the same axial pencil made by the transversal planes Ou, $O'u'$, the axial pencil namely which is composed of the planes $OO'A$, $OO'B$, $OO'C$, , and which has for axis the straight line OO' This is the general case of two flat pencils in perspective, they have not the same centre and they lie in different planes, at the same time, they project the same range and are sections of the same axial pencil. There are two exceptional cases (1) If we project the row u from two centres O and O' lying in the same plane with u, then the two resulting flat pencils lie in the same plane and are consequently no longer sections of an axial pencil, (2) If an axial pencil is cut by two transversal planes which pass through a common point O on the axis, we obtain two flat pencils which have the same centre O and which consequently no longer project the same range]

Two axial pencils, if they project, from two different centres, the same flat pencil

A range and a flat pencil, a range and an axial pencil, or *a flat pencil and an axial pencil*, if the first is a section of the second

Two plane figures, if they are plane sections of the sa
sheaf

Two sheaves, if they project, from two different centres,
same plane figure

A plane figure and a sheaf, if the former is a section of
latter

From the definition of Art 40 it follows at once that
(one-dimensional) forms which are in perspective are also p
jectively related, but two projective forms are not in gene
in perspective position

43 Two figures in homology are merely two project
plane figures superposed one upon the other, in a particu
position; for by Art 21 two homological figures may alw
be regarded (and this in an infinite number of ways) as p
jections of one and the same third figure

If two projective plane figures are superposed one upon
other in such a manner that the straight line connecting a
pair of corresponding points may pass through a fixed poi
or, again, in such a manner that any pair of correspond
straight lines may intersect on a fixed straight line, then
two figures are in homology (Arts 19, 20)

In two homological figures, two corresponding ranges are
perspective (and therefore of course are projectively relate
and the same is the case with regard to two correspond
pencils

✓ 44 THEOREM *Two one-dimensional geometric forms, each c
sisting of three elements, are always projective*

To prove this, we notice in the first place that it
enough to consider the case of two ranges ABC, $A'B'C'$,
if one of the given forms is a pencil, flat or axial, we m
substitute for it one of its sections by a transversal

(1) If the two straight lines ABC, $A'B'C'$ lie in differ
planes, join AA', BB', CC', and cut these straight li
by a transversal s* Then the two given forms are s
to be simply two sections of the axial pencil sAA', sB
sCC'

(2) If the two straight lines lie in the same plane (Fig
join AA', and take on this straight line any two points, S,

* To do this, we have only to draw through any point of AA' a straight

draw SB, $S'B'$ to cut in B'', and SC, $S'C'$ to cut in C'', and $B''C''$, cutting SS' in A''. Then $A'B'C'$ may be derived f

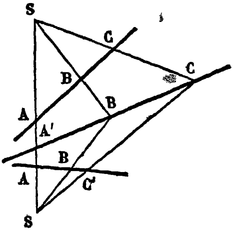


Fig 21

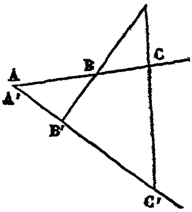


Fig 22.

ABC by two projections, viz we first project ABC fro into $A''B''C''$, and then $A''B''C''$ from S' into $A'B'C'$.

(3) In the case where the two points A and A' coincide (22), the two given forms are directly in perspective, the ce of perspective is the point where BB' and CC' intersect

(4) If the two sets of points ABC, $A'B'C'$ lie on the same stra line (Fig 23), it is only necessary to project one of them $A'B'C'$ on to another straight line $A_1B_1C_1$ (from any centre O), then let any two centres S and S_1 be taken (as in Fig 21) on AA_1, and let the straight line $A''B''C''$ be constructed in the manner already shown in case (2) Then $A'B'C'$ may be derived from ABC by three projections, viz we first project ABC from S into $A''B''C''$, then $A''B''C''$ from S_1 into $A_1B_1C_1$, and lastly $A_1B_1C_1$ from O into $A'B'C'$

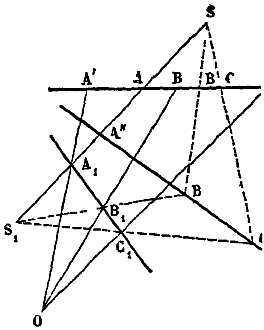


Fig 23

(5) If A coincides with A', and B with B', we may n use of a centre S and two transversals s_1, s_2 drawn throug in the plane $SABCC'$ If the triad ABC be projected fro upon s_1 (giving $A_1B_1C_1$), and the triad $A'B'C'$ be proje from S upon s_2 (giving $A_2B_2C_2$), then the triads $A_1B_1C_1$ $A_2B_2C_2$ will be in perspective, because A_1 coincides with A the point AA')

In every case, then, it has been shown that the tı

ABC, $A'B'C'$ can be derived from each other by a finit number of projections and sections, therefore by Art 4 they are projective

As a particular case, ABC must be projective with BAC, fo example In order actually to project one of these triads int the other, take (Fig 24) any two points L and N collinea with C Join AL, BN, meetin in K, and BL, AN, meeting i M Then BAC can be derive from ABC by first projectin ABC from K into LNC, and the LNC from M into BAC

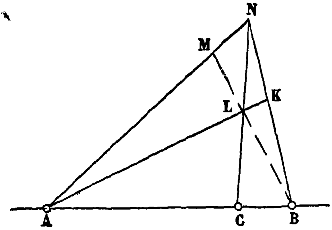


Fig 24.

In order to project ABC in BCA, we might first proje ABC into BAC, and then BA into BCA

45 THEOREM *Any one-dimensional geometric form, consisti of four elements, is projective with any of the forms derived from by interchanging the elements in pairs For instance, ABCD projective with BADC*

Let A, B, C, D be four given points (Fig 25), and l $EFGD$ be a projection of these poin from a centre M on a straight li DF passing through D If AF, C meet in N, then $MNGC$ will be projection of $EFGD$ from centre and $BADC$ a projection of MN(from centre F, therefore (Ar 40, 41) the form $BADC$ is pr jective with $ABCD$ In a similar manner it can be show that $CDAB$ and $DCBA$ are projective with $ABCD$ *

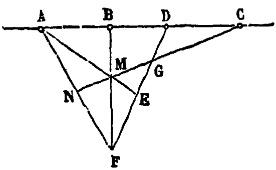


Fig 25

From this it follows for example that if a flat pencil $abcd$ projective with a range $ABCD$, then it is projective also wi $BADC$, with $CDAB$, and with $DCBA$, *i e if two geometric form each consisting of four elements, are projectively related, then elements of the one can be made to correspond respectively to elements of the other in four different ways*

* STAUDT, *Geometrie der Lage*, Art 59

CHAPTER VIII

HARMONIC FORMS

46 THEOREM*

Given three points A, B, C on a straight line s, if a complete quadrangle $(KLMN)$ be constructed (in any plane through s) in such a manner that two opposite sides (KL, MN) meet in A, two other opposite sides (KN, ML) meet in B, and the fifth side (LN) passes through C, then the sixth side (KM) will cut the straight line s in a point D which is determined by the three given points, *i e* it does not change its position, in whatever manner the arbitrary elements of the quadrangle are made to vary (Fig 26)

Given in a plane three str lines a, b, c which meet in a S, if a complete quadril $(klmn)$ be constructed in su manner that two opposite ve (kl, mn) lie on a, two other site vertices (kn, ml) lie on the fifth vertex (nl) lies then the sixth vertex $(km$ lie on a straight line d passes through S, and wh determinate, *i e* it doe change its position, in wh manner the arbitrary eleme the quadrilateral are ma vary (Fig 27)

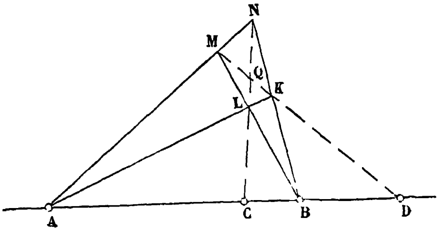


Fig 26

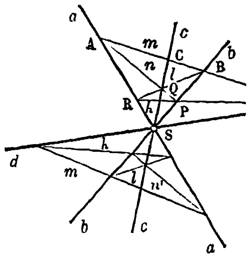


Fig 27

For if a second complete quadrangle $(K'L'M'N')$ be con

For if a second cc quadrilateral $(k'l'm'n')$ l

* STAUDT, *loc cit*, Art 93

structed (either in the same plane, or in any other plane through s), which satisfies the prescribed conditions, then the two quadrangles will have five pairs of corresponding sides which meet on the given straight line, therefore the sixth pair will also meet on the same line (Art 36, No 5, left)

From this it follows that if the first quadrangle be kept fixed while the second is made to vary in every possible way, the point D will remain fixed, which proves the theorem

The four points $ABCD$ are called *harmonic*, or we may say that the group or the *geometric form* constituted by these four points is a *harmonic* one, or that $ABCD$ form a *harmonic range* Or again *Four points $ABCD$ of a straight line, taken in this order, are called harmonic, if it is possible to construct a complete quadrangle such that two opposite sides pass through A, two other opposite sides through B, the fifth side through C, and the sixth through D* It follows from the preceding theorem that when such a quadrangle exists, *i e* when the form $ABCD$ is harmonic, it is possible to construct an infinite number of other quadrangles satisfying the same conditions It further follows that, given three points ABC of a range (and also the order in which they are to be taken), the fourth point D, which makes with them a harmonic form, is *determinate* and *unique*, and is found by the construction of one of the quadrangles (see below, Art 58)

structed which satisfies the prescribed conditions, then the two quadrilaterals will have five pairs of corresponding vertices collinear respectively with the given point, therefore the sixth pair will also lie in a straight line passing through the same point (Art 36, No 5, right)

From this it follows that if the first quadrilateral be kept fixed while the second is made to vary in every possible way, the straight line d will remain fixed, which proves the theorem

The four straight lines or rays $abcd$ are called *harmonic*, or we may say that the group or the *geometric form* constituted by these four lines is a *harmonic* one, or that $abcd$ form a *harmonic pencil* Or again *Four rays $abcd$ of a pencil, taken in this order, are called harmonic, if it is possible to construct a complete quadrilateral such that two opposite vertices lie on a, two other opposite vertices on b, the fifth vertex on c, and the sixth on d* It follows from the preceding theorem that when such a quadrilateral exists, *i e* when the form $abcd$ is harmonic, it is possible to construct an infinite number of other quadrilaterals satisfying the same conditions It further follows that given three rays abc of a pencil (and also the order in which they are to be taken), the fourth ray d, which makes with them a harmonic form, is *determinate* and *unique*, and is found by the construction of one of the quadrilaterals (see below, Art 58)

47 *If from any point S the harmonic range ABCD be proj upon any other straight line, its projection A′B′C′D′ will also harmonic range* (Fig 28)

Imagine two planes drawn one through each of the stra lines AB, $A'B'$, and suppose that in the first of these pl is constructed a complete quadrangle of which two opposite sides meet in A, two other opposite sides meet in B, and a fifth side passes through C, then the sixth side will pass through D (Art 46), since by hypothesis $ABCD$ is a harmonic range Now project this quadrangle from the point S on to the second plane, a new quadrangle is obtained of which two opposite meet in A', two other opposite sides meet in B', and w fifth and sixth sides pass respectively through C' and therefore $A'B'C'D'$ is a harmonic range.

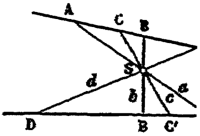


Fig 28

48 An examination of Fig 27 will show that the harm pencil *abcd* is cut by any transversal whatever in a monic range For let S be the centre of the pencil and any transversal, in a take any point R, join R to D b straight line k and to B by the straight line l, and join kb or P by the straight line n As *abcd* is a harmonic p and five vertices of the complete quadrilateral $klmn$ lie on and d, the sixth vertex ln or Q must lie on the fourth Then from the complete quadrangle $PQRS$ it is clear $ABCD$ is a harmonic range

Conversely, if the harmonic range $ABCD$ (Fig 27) be g and any centre whatever of projection S be taken, then four projecting rays $S(A, B, C, D)$ will form a harm pencil

For draw through A any straight line to cut SB in P SC in Q, and join BQ, cutting AS in R The quadrangle P is such that two opposite sides meet in A, two other opp sides in B, and the fifth side passes through C, consequ the sixth side must pass through D (Art 46, left), sinc hypothesis the range $ABCD$ is harmonic But then we a complete quadrilateral $klmn$ which has two opposite ve A and R lying on SA, two other opposite vertices B and SB, a fifth vertex Q on SC, and the sixth D on SD, then

(Art. 46, right) the four straight lines which project the range $ABCD$ from S are harmonic We may therefore enunciate the following proposition

A harmonic pencil is cut by any transversal whatever in a harmonic range, and, conversely, the rays which project a harmonic range from any centre whatever form a harmonic pencil

Corollary In two homological figures, to a range of four harmonic points corresponds a range of four harmonic points, and to a pencil of four harmonic rays corresponds a pencil of four harmonic rays

49 The theorem on the right in Art 46 is correlative to that on the left in the same Article In this latter theorem all the quadrangles are supposed to lie in the same plane, but from the preceding considerations it is clear that the theorem is still true and may be proved in the same manner, if the quadrangles are drawn in different planes

Considering accordingly this latter theorem (Art 46, left) as a proposition in the Geometry of space, the theorem correlative to it will be the following

If three planes α, β, γ all pass through one straight line s, and if a complete four-flat (see Art 37) $\kappa\lambda\mu\nu$ *be constructed, of which two opposite edges $\kappa\lambda$, $\mu\nu$ lie in the plane α, two other opposite edges $\kappa\nu$, $\lambda\mu$ lie in the plane β, and the edge $\lambda\nu$ lies in the plane γ, then the sixth edge $\kappa\mu$ will always lie in a fixed plane δ (passing through s), which does not change, in whatever manner the arbitrary elements of the four-flat be made to vary*

For if we construct (taking either the same vertex or any other lying on s) another complete four-flat which satisfies the prescribed conditions, the two four-flats will have five pairs of corresponding edges lying in planes which all pass through the same straight line s, therefore (Art 37, left) the sixth pair also will lie in a plane which passes through s The four planes, α, β, γ, δ are termed *harmonic planes*, or we may say that the group or the *geometric form* constituted by them is *harmonic*, or again that they form a *harmonic (axial) pencil*

50 If a complete four-flat $\kappa\lambda\mu\nu$ be cut by any plane not passing through the vertex of the pencil, a complete quadrilateral is obtained, and the same transversal plane cuts the planes α, β, γ, δ in four rays of a flat pencil of which the first

two rays contain each a pair of vertices of the quadrilate while the other two pass each through one of the remain vertices Consequently (Art. 46, right) an axial pencil of f harmonic planes is cut by any transversal plane in a flat pe of four harmonic rays

Similarly, if the harmonic axial pencil of four pla $\alpha, \beta, \gamma, \delta$ is cut by any transversal line in four points $A, B, C,$ these form a harmonic range For if through the transve line a plane be drawn, it will cut the planes $\alpha, \beta, \gamma, \delta$ in f straight lines a, b, c, d This group of straight lines is h monic, by what has just been proved, but $ABCD$ is a sect of the flat pencil a, b, c, d, consequently (Art. 48) the f points A, B, C, D are harmonic. Conversely, if four poi forming a harmonic range be projected from an axis, or if f rays forming a harmonic pencil be projected from a point, resulting axial pencil is harmonic

51 If then we include under the title of *harmonic form* group of four harmonic points (the harmonic range), the gr of four harmonic rays (the harmonic flat pencil), and group of four harmonic planes (the harmonic axial pencil), may enunciate the theorem

Every projection or section of a harmonic form is itself a harm form or,

Every form which is projective with a harmonic form is i harmonic

Conversely, *two harmonic forms are always projective with another*

To prove this proposition, it is enough to consider groups each of four harmonic points, for if one of the fo were a pencil we should obtain four harmonic points cutting it by a transversal Let then $ABCD$, $A'B'C'D'$ be harmonic ranges, and project ABC into $A'B'C'$ in the man explained in Art 44, the same operations (projections sections) which serve to derive $A'B'C'$ from ABC will give D a point D_1, from which it follows that the range $A'B'C$ will be harmonic, since the range $ABCD$ is harmonic $A'B'C'D'$ are also four harmonic points, by hypothesis, th fore D_1 must coincide with D', since the three points A determine uniquely the fourth point which forms with the harmonic range (Art 46, left)

We may add here a consequence of the definitions given Arts 49 and 50

The form which is correlative to a harmonic form is itse harmonic

52 If a , b , c , d are rays of a pencil (Fig 28), then a and are said to be *separated* by c and d, when a straight line pas ing through the centre of the pencil, and rotating so as come into coincidence with each of the rays in turn, cann pass from a to b without coinciding with one and only one the two other rays c and d* The same definition applies the case of four planes of a pencil, and to that of four points a range (Fig 26), only it must be granted that we may pa from a point A to a point B in two different ways, either l describing the finite segment AB or the infinite segment whi begins at A, passes through the point at infinity, and ends at

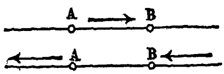


Fig 29

This definition premised, the follov ing property may be enunciated as once evident Four elements of a on dimensional geometric form (*i e* fo points of a range, four rays of pencil, &c) can always be so divided into two pairs th one pair is separated by the other, and this can be done one way only In Fig 26, for example, the two pairs whi separate one another are AB , CD , and if $A'B'C'D'$ is a for projective with $ABCD$, the pair $A'B'$ will be separated by t pair $C'D'$, for the operations of projection and section do n change the relative position of the elements

53 Let now $ABCD$ (Fig 30) be four harmonic points, *i e* fo points obtained by the construction of Art 46, left Th allows us to draw in an infinite number of ways a comple quadrangle of which A and B are two diagonal poin (Art 36, No 2, left), while the other two opposite sides pa through C and D It is only necessary to state this co struction in order to see that the two points A and B a precisely similar in their relation to the system, and that t same is true with regard to C and D It follows from th that if $ABCD$ is a harmonic range, then $BACD$, $ABDC$, BAL which are obtained by permuting the letters A and B or and D, or both at the same time, are harmonic ranges al

* a and b, c and d, may also be termed *alternate* pairs of rays

Consequently (Art 51) the harmonic range $ABCD$ for examp is projective with $BACD$, i e we can pass from one range to th other by a finite number of projections and sections. In fa if the range $ABCD$ be projected from K on CQ, we obtain th range $LNCQ$, which when projected from M on AB giv $BACD$.

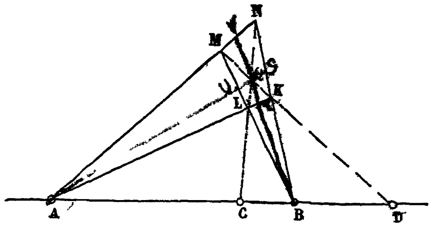


Fig 30

54 *If A, B, C, D are four harmonic points, then A and B a necessarily separated by C and D*

For if (Fig 30) the group $ABCD$ be projected on the straig line KM, first from the centre L and then from the centre the projections are $KMQD$ and $MKQD$ respectively Now, already stated in Art 52, the operations of projection a section do not change the relative position of the elemen the group If therefore K and Q were separated by M and then also M and Q must be separated by K and D, which impossible The only possible arrangement is that K a M should be separated by Q and D, and therefore A and separated by C and D

$(ABCD)^H \supset (CDAB)^H$

55 Let the straight lines AQ, BQ be drawn (Fig 31), t former meeting MB in U and NB in S, while the latter meets KL in T and MN in V The complete quadrangle $LTQU$ has two opposite sides meeting in A, two other opposite sides meeting in B, and a fifth side (LQ or LN) passes through C, therefore the sixth side UT will pass through D (A 46) In like manner the sixth side VS of the complete quadrang $NVQS$ must pass through D, and the sixth sides of the co plete quadrangles $KSQT$, $MUQV$ through C We have thu quadrangle $STUV$ two of whose opposite sides meet in C, t

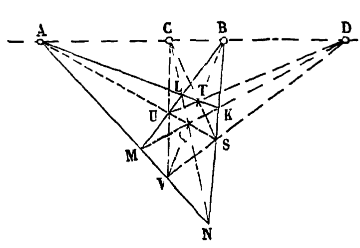


Fig 31

other opposite sides in D, while the fifth and sixth sides p respectively through A and B This shows that the relat to which the points C and D are subject (Art 53) is the sa as the relation to which the points A and B are subject, in other words, that the pair A, B may be interchan with the pair C, D Accordingly, if $ABCD$ is a harmo range, then not only the ranges $BACD$, $ABDC$, $BADC$, also $CDAB$, $DCAB$, $CDBA$, $DCBA$ are harmonic *

The points A and B are termed *conjugate* points, as also C and D Or either pair are said to be *harmonic conjug* with respect to the other The points A and B are said to *harmonically separated* by the points C and D, or the point and D to be harmonically separated by A and B We m also say that the segment AB is divided harmonically by segment CD, or that the segment CD is divided harmonica by AB If two points A and B (Fig 30) are separated h monically by the points C and D in which the straight l AB is cut by two straight lines QC and QD, we may also that the segment AB is divided harmonically by the strai lines QC, QD, or by the point C and the straight line QD, & and that the straight lines QC, QD are separated harmonica by the points A, B, &c

Analogous properties and expressions exist in the case four harmonic rays or four harmonic planes

[*Note*—In future, whenever mention is made of the harm system $ABCD$, it is always to be understood that A and B, C and D, conjugate pairs, it being at the same time remembered that (Art A and B, C and D, are necessarily alternate pairs of points]

56 The following theorem is another consequence of proposition of Art 46, left

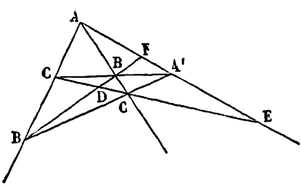


Fig 32

In a complete quadrilateral, e diagonal is divided harmonicall the other two †

Let A and A', B and B' and C' be the pairs of oppo vertices of a complete quad lateral (Fig 32), and let diagonal AA' be cut by the other diagonals BB' and CC' i

* REYE *Geometrie der Lage* (Hanover, 1866), vol 1 p 34

† CARNOT [illegible] (Par [illegible]) Art 22

and E respectively Consider now the complete quadra $BB'CC'$, one pair of its opposite sides meet in A, ano such pair in A', a fifth side passes through E, the s through F The points A, A' are therefore harmonic separated by F and E Similarly a consideration of the complete quadrangles $CC'AA'$ and $AA'BB'$ will show B, B' are harmonically separated by F and D, and C, C D and E

57 In the complete quadrangle $BB'CC'$ the diagonal po are A, A', and D, also since the range $BB'FD$ is harmoni too is the pencil of four rays which project it from A (Art. therefore

In a complete quadrangle, any two sides which meet in a diag point are divided harmonically by the two other diagonal points

This theorem is however merely the correlative (in acc ance with the principle of Duality in plane Geometry) of proved in the preceding Article

58 The theorems of Art 46 can be at once applied to solution, by means of the ruler only, of the following blems

Given three points of a harmonic range, to find the fourth

Solution Let A, B, C (Fig 33) be the given points (lying on a given straight line) and let

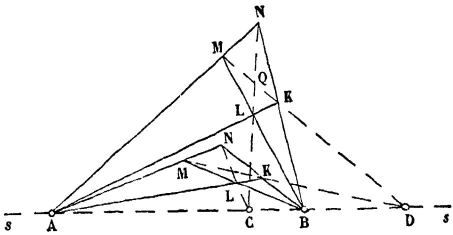


Fig 33

A and B be conjugate to each other Draw any two straight lines through A, and a third through C to cut these in L and

Given three rays of a harmonic pencil, to construct fourth

Solution Let a, b, c 34) be the given rays (lyi one plane and passing throu

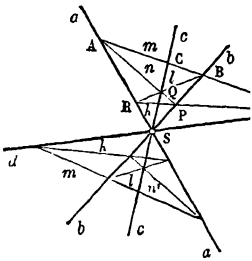


Fig 34

given centre S), and let a be conjugate to each Through any point Q lying draw any two straight lin

N respectively Join BL cutting AN in M, and BN cutting AL in K, then if KM be joined it will cut the given straight line in the required point D, conjugate to C*.

cut a in A and R, and b i and B, respectively Join AB RP, these will cut in a point the line joining which to S is required ray d, conjugate to c

59. In the problem of Art 58, left, let C lie midway betwee and B We can, in the solution, so arrange the arbitrary elem that the points K and M shall move to infinity, to effect this we must struct (Fig 35) a parallelogram AL on AB as diagonal, then since the o diagonal LN passes through C, the p D will lie at infinity

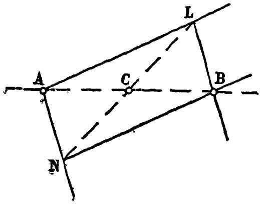


Fig 35.

If, conversely, the points A, B, D given, of which the third point D at infinity, we may again construc parallelogram $ALBN$ on AB as diagonal, then the fourth point the conjugate of D, must be the point where LN meets the gi straight line that is, it must be the middle point of AB Therefo

If in a harmonic range AB the point C lies midway between two conjugates A and B, then the fo point D lies at an infinite dista and conversely, if one of the point lies at infinity, its conjugate C is point midway between the two oth A and B

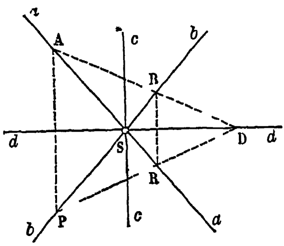


Fig 36

60 In the problem of Art right, let c be the bisector of angle between a and b (Fig 36) Q be taken at infinity on c, the segments AB, PR become equa one another and lie between the pa AP, BR, consequently the ray d wi perpendicular to c, *i e given a harm pencil of four rays, abcd if on them c bisect the angle between the conjugates a and b, the fourth ra will be at right angles to c*

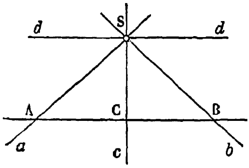


Fig 37

Conversely *if in a harmonic pencil* (Fig 37) *two conjugate rays c, d are at right angles, then they ar bisectors, internal and external, of the angle between the other two a, b*

* De la Hire, *Sectiones Conicae* (Parisiis, 1685), lib i, prop 20

For if the pencil be cut by a transversal AB drawn parallel t the section $ABCD$ will be a harmonic range (Art 48), and a lies at infinity, C must lie midway between A and B (Art. 59), co quently, if S be the centre of the pencil, ASB is an isosceles tria and SC the bisector of its vertical angle

CHAPTER IX

ANHARMONIC RATIOS

61 GEOMETRICAL propositions divide themselves into t
classes Those of the one class are either immediately c
cerned with the magnitude of figures, as Euc I 47, or tl
involve more or less directly the idea of quantity or measu
ment, as *e g* Euc I 12 Such propositions are called *metri*
The other class of propositions relate merely to the posit
of the figures with which they deal, and the idea of quant
does not enter into them at all Such propositions are cal
descriptive Most of the propositions in Euclid's *Elements*
metrical, and it is not easy to find among them an exampl
a purely descriptive theorem Prop 2, Book XI, may se
as an instance of one Projective Geometry on the ot
hand, dealing with projective properties (*i e* such as are
altered by projection), is chiefly concerned with descript
properties of figures In fact, since the magnitude of a g
metric figure is altered by projection, metrical properties
as a rule not projective But there is one important class
metrical properties (anharmonic properties) which are p
jective, and the discussion of which therefore finds a place
the Projective Geometry To these we proceed, but it
necessary first to establish certain fundamental notions

62 Consider a straight line, a point may move along it
two different directions, one of which is opposite to the otl
Let it be agreed to call one of these the positive direction, a
the other the negative direction Let A and B be two poi
on the straight line, and let it be further agreed to repres
by the expression AB the length of the segment compri
between A and B, taken as a positive or as a negative num
of units according as the direction is positive or negative
which a point must move in order to describe the segme

this point starting from A (the first letter of the expres
AB) and ending at B

In consequence of this convention, which is termed the *of signs*, the two expressions AB, BA are quantities which equal in magnitude but opposite in sign, so that $BA = -A$

$$AB + BA = 0$$

Now let A, B, C be three points lying on a straight l If C lies between A and B (Fig 38 *a*),

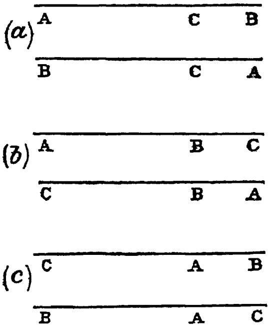


Fig 38

we have $$AB = AC + CB,$$
whence $$-CB - AC + AB = 0,$$
or $$BC + CA + AB = 0$$

Again, if B lies between A and C (Fig 38 *b*),

$$AC = AB + BC,$$
whence $$BC - AC + AB = 0,$$
or $$BC + CA + AB = 0$$

Lastly, if A lies between B and C (Fig 38 *c*),

$$CB = CA + AB,$$
whence $$-CB + CA + AB = 0,$$
or $$BC + CA + AB = 0$$

Accordingly

If A, B, C are three collinear points, then whatever their re positions may be, the identity

$$BC + CA + AB = 0$$

always holds good

From this identity may be deduced an expression fo distance between two points A and B in terms of the dist

of these points from an origin O chosen arbitrarily on straight line which joins them

For since $$OA + AB + BO = 0,$$
$$AB = OB - OA,$$
or again, $$AB = AO + OB*$$

The results (1) and (2) may be extended, they are in particular cases of the following general proposition

If $A_1, A_2, \quad A_n$ be n collinear points, then
$$A_1A_2 + A_2A_3 + \quad + A_{n-1}A_n + A_nA_1 = 0,$$
the truth of which follows at once from (3), since the ex] sion on the left hand is equal to
$$(OA_2 - OA_1) + (OA_3 - OA_2) + \quad + (OA_1 - OA_n),$$
which vanishes

Another useful result is that if A, B, C, D be four colli points,
$$BC \quad AD + CA \quad BD + AB \quad CD = 0$$

This again follows from (3), since the left-hand side
$$= (DC - DB)AD + \quad +$$
$$= 0$$

Many other relations of a similar kind between segm might be proved, but they are not necessary for our pur] We will give only one more, viz

If A, B, C, O be any four collinear points, then
$$OA^2 \quad BC + OB^2 \quad CA + OC^2 \quad AB = -BC \quad CA \quad AB$$

For by (3) the left-hand side is equal to
$$(OA^2 - OC^2)BC + (OB^2 - OC^2)CA$$
$$= CA(OA + OC)BC + CB(OB + OC)C$$
$$= BC \quad CA(OA - OB)$$
$$= -BC \quad CA \quad AB$$

It may be noticed that this last theorem is true even if not lie on the straight line ABC, but be any point what For if a perpendicular OO' be let fall on ABC,
$$OA^2 \quad BC + OB^2 \quad CA + OC^2 \quad AB$$
$$= (OO'^2 + O'A^2)BC + \quad +$$
$$= O'A^2 \quad BC + O'B^2 \quad CA + O'C^2 \quad AB$$
$$+ OO'^2(BC + CA +$$
$$= -BC \quad CA \quad AB,$$
by what has just been proved

63 Consider now Fig 39, which represents the proje

* MÖBIUS, *Barycentrische Calcul*, § 1

from a centre S of the points of a straight line a on to ano
straight line a', let us examine the relation which e
between the lengths of two corresponding segments AB, A

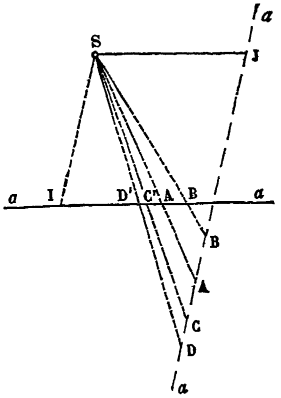


Fig 39

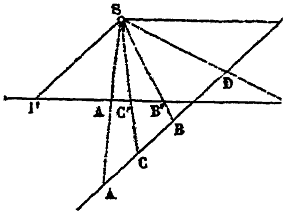


Fig 40.

From the similar triangles SAJ, $A'SI'$

$$JA \quad JS \quad I'S \quad I'A',*$$

so from the similar triangles SBJ, $B'SI'$,

$$JB \quad JS \quad I'S \quad I'B',$$

$$JA \quad I'A' = JB \quad I'B' = JS \quad I'S,$$

i e the rectangle $JA \quad I'A'$ has a constant value for all of corresponding points A and A'

If the constant $JS \quad I'S$ be denoted by k, we have

$$I'A' = \frac{k}{JA}, \quad I'B' = \frac{k}{JB},$$

therefore by subtraction,

$$I'B' - I'A' = \frac{k(JA - JB)}{JA \quad JB}$$

But $I'B' - I'A' = A'B'$, and $JA - JB = BA = -AB$,

$$A'B' = \frac{-k}{JA \quad JB} \quad AB$$

If we consider four points A, B, C, D (Fig 40)
straight line a and their four projections, A', B', C', D
obtain, in a similar manner,

* We suppose that in all equations involving segments the rule of observed See MÖBIUS, *Baryc Calcul*, § 1, TOWNSEND, *Modern Ge* chapter v

$$A'C' = \frac{-k}{JA \cdot JC}\,AC,$$

$$B'C' = \frac{-k}{JB \cdot JC}\,BC,$$

$$A'D' = \frac{-k}{JA \cdot JD}\,AD,$$

$$B'D' = \frac{-k}{JB \cdot JD}\,BD,$$

whence by division

$$\frac{A'C'}{B'C'} : \frac{A'D'}{B'D'} = \frac{AC}{BC} : \frac{AD}{BD}.$$

This last equation, which has been proved for the case of projection from a centre S, holds also for the case where $ABCD$ and $A'B'C'D'$ are the intersections of two transversal lines s and s' (not lying in the same plane) with four planes $\alpha, \beta, \gamma, \delta$ which all pass through one straight line u, in other words, when $A'B'C'D'$ is a projection of $ABCD$ made from an axis u (Art 4). For let the four planes $\alpha, \beta, \gamma, \delta$ be cut in A'', B'', C'', D'' respectively by a straight line s'' which meets s and s'. The straight lines AA'', BB'', CC'', DD'' are the intersections of the planes $\alpha, \beta, \gamma, \delta$ respectively by the plane ss'', and therefore meet in a point S, that namely in which the plane ss'' is cut by the axis u. So also $A'A''$, $B'B''$, $C'C''$, $D'D''$ are four straight lines lying in the plane $s's''$ and meeting in a point S' of the axis u (that namely in which the plane $s's''$ is cut by the axis u). Therefore $A''B''C''D''$ is a projection of $ABCD$ from centre S and a projection $A'B'C'D'$ from centre S', so that

$$\frac{A''C''}{B''C''} : \frac{A''D''}{B''D''} = \frac{AC}{BC} : \frac{AD}{BD} = \frac{A'C'}{B'C'} : \frac{A'D'}{B'D'}.$$

The number

$$\frac{AC}{BC} : \frac{AD}{BD}$$

is called the *anharmonic ratio* of the four collinear points A, B, C, D. The result obtained above may therefore be expressed as follows

The anharmonic ratio of four collinear points is unaltered by any projection whatever. *

* PAPPUS *Mathematicae Collectiones*, book vii prop 129 (ed Hultsch, Berlin, 1877, vol ii p 871)

Or again

If two ranges, each of four points, are projective, they have same anharmonic ratio, or, as we may say, *are equianharmonic*

64 Dividing one by the other the expressions for $A'C'$ $B'C'$, we have

$$\frac{A'C'}{B'C'} = \frac{AC}{BC}\ \frac{AJ}{BJ}$$

In this equation the right-hand member is the anharm ratio of the four points A, B, C, J, consequently the left-h member must be the anharmonic ratio of A', B', C', J', *the anharmonic ratio of four points A', B', C', J', of which the lies at infinity, is merely the simple ratio $A'C'$ $B'C'$*

This may also be seen by observing that if A' and remain fixed while D' moves off to infinity on the line A then

$$\text{limiting value of } \frac{A'D'}{B'D'} = 1,$$

$$\text{limiting value of } \frac{A'C'}{B'C'}\ \frac{A'D'}{B'D'} = \frac{A'C'}{B'C'}$$

Similarly, on the same supposition,

$$\text{limiting value of } \frac{A'D'}{B'D'}\ \frac{A'C'}{B'C'} = \frac{B'C'}{A'C'},$$

i e the anharmonic ratio of the four points A', B', D', C', of the third lies at infinity, is equal to the simple ratio $B'C'$ $A'C$

65 From this results the solution of the following

PROBLEM —*Given three collinear points A, B, C, to find a f D so that the anharmonic ratio of the range $ABCD$ may number λ given in sign and magnitude* (Fig 41)

Solution —Draw any transversal through C, and tak it two points A', B' such that the ratio CA' CB' is equal to λ 1, the given value of the anharmonic ratio, the two points A' and B' lying on the same or on opposite sides of C according as λ is positive or negative Join AA', BB', meeting in S, the straight line through S parallel to $A'B'$ will cut in the point D required † For if D' be the point at infinit

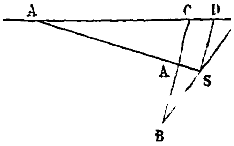


Fig 41

* TOWNSEND *Modern Geometry*, Art 278

† CHASLES, *Géométrie supérieure* (Paris, 185), p 10

$A'B'$, and we consider $ABCD$ as a projection of $A'B'C'D'$ (C' coincides with C) from the centre S, then the anharmonic ratio of $ABCD$ is equal to that of $A'B'C'D'$, that is, to the simple ratio $A'C' : B'C'$ or λ.

The above is simply the graphical solution of the equation

$$\frac{AC}{BC} : \frac{AD}{BD} = \lambda,$$

or

$$\frac{AD}{BD} = \frac{AC}{BC} : \lambda = \mu,$$

or in other words of the problem

Given two points A and B, to find a point D collinear with them such that the ratio of the segments AD, BD to one another may be equal to a number given in sign and magnitude

As only one such point D can be found, the proposed problem admits of only one solution, this is also clear from the construction given, since only one line can be drawn through S parallel to $A'B'$ Consequently there cannot be two different points D and D_1 such that $ABCD$ and $ABCD_1$ have the same anharmonic ratio Or

If the groups $ABCD$, $ABCD_1$ are equianharmonic, the point D_1 must coincide with D

V

66 THEOREM (Converse to that of Art 63) *If two ranges $ABCD$, $A'B'C'D'$, each of four points, are equianharmonic, they are projective with one another*

For (by Art 44) we can always pass from the triad ABC to the triad $A'B'C'$ by a finite number of projections and sections, let D'' be the point which these operations give as corresponding to D Then the anharmonic ratio of $A'B'C'D''$ will be equal to that of $ABCD$, and consequently to that of $A'B'C'D'$, whence it follows that D'' coincides with D', and that the ranges $ABCD$, $A'B'C'D'$ are projective with one another

67 It follows then from Arts 63 and 66 that the necessary and sufficient condition that two ranges $ABCD$, $A'B'C'D'$ consisting each of four points, should be projective, is the equality (in sign and magnitude) of their anharmonic ratios

The anharmonic ratio of four points $ABCD$ is denoted by the symbol $(ABCD)$*, accordingly the projectivity of two forms $ABCD$ and $A'B'C'D'$ is expressed by the equation

$$(ABCD) = (A'B'C'D')$$

* MÖBIUS, *Barycentrische Calcul*, § 183

From what has been proved it is seen that if two pen each consisting of four rays or four planes are cut by any t transversals in $ABCD$ and $A'B'C'D'$ respectively, the equat $(ABCD) = (A'B'C'D')$ is the necessary and sufficient condit that the two pencils should be projective with one another

The *anharmonic ratio of a pencil of four rays* a, b, c, d *four planes* $\alpha, \beta, \gamma, \delta$ may now be defined as the const anharmonic ratio of the four points in which the four eleme of the pencil are cut by any transversal, and may be denc by $(abcd)$ or $(\alpha\beta\gamma\delta)$

This done, we can enunciate the general theorem

If two one-dimensional geometric forms, consisting each of elements, are projective, they are equianharmonic, and if they equianharmonic, they are projective

68 Since two harmonic forms are always projecti related (Art 51), the preceding theorem leads to the c clusion that the anharmonic ratio of four harmonic elem is a *constant number* For if $ABCD$ is a harmonic sys $BACD$ is also a harmonic system (Art 53), and the systems $ACBD$ and $BCAD$ are projectively related*, thus

$$(ACBD) = (BCAD),$$

since AI = BC = CB

i e

$$\frac{AB}{CB} \cdot \frac{AD}{CD} = \frac{BA}{CA} \cdot \frac{BD}{CD},$$

whence

$$\frac{AC}{BC} \cdot \frac{AD}{BD} = -1,$$

i e

$$(ABCD) = -1,$$

therefore *the anharmonic ratio of four harmonic elements is to* -1 †

69 The equation $(ABCD) = -1$, or

$$\frac{AC}{BC} + \frac{AD}{BD} = 0,$$

which expresses that the range $ABCD$ is harmonic, may be put two other remarkable forms

Since $AD = CD - CA$ (Art 62) and $BD = CD - CB$, the equ (1) gives

$$CA(CD - CB) + CB(CD - CA) = 0$$

or

$$\frac{1}{CD} = \tfrac{1}{2}\left(\frac{1}{CA} + \frac{1}{CB}\right),$$

* In Fig 30 $ACBD$ may be projected (from K on NC) into $LCNQ$, and $LCNQ$ may be projected (from M on AD) into $LCAD$

† Mobius, *loc cit*, p 269

i e. CD is the harmonic mean between CA and CB, a formula wl determines the point D when A, B, C are given

Again, if O is the middle point of the segment CD, so that we h $OD = CO = -OC$, then

$$AC = OC - OA, \qquad AD = OD - OA = -(OC + OA),$$
$$BC = OC - OB, \qquad BD = -(OC + OB)$$

Substituting these values in (1) or in

$$\frac{AC}{AD} + \frac{BC}{BD} = 0,$$

we have
$$\frac{OC - OA}{OC + OA} = \frac{OB - OC}{OB + OC},$$
$$\frac{OC}{OA} = \frac{OB}{OC},$$

or
$$OC^2 = OA \cdot OB,$$

i.e half the segment CD is a mean proportional between the dista of A and B from the middle point of CD

The equation (3) shows that the segments OA and OB must h the same sign, and that O therefore can never lie between A and B

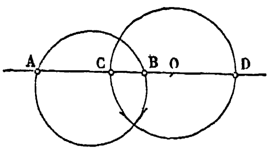


Fig 42

If now a circle be drawn to p through A and B (Fig 42), O will outside the circle, and OC will be length of the tangent from O to (Euc III 37) The circle on CD diameter will therefore cut the f circle (and *all* circles through A and orthogonally Conversely, if two circles cut each other orthogon they will cut any diameter of one of them in two pairs of harm points †.

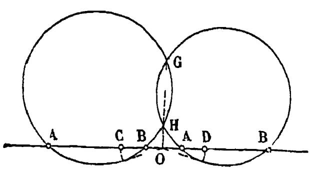


Fig 43

70 The same formula (3) g the solution of the following p blem

Given two collinear segments and $A'B'$, to determine ano segment CD which shall divide of them harmonically (Figs 43,

Take any point G not lying the common base AB', and draw the circles GAB, $GA'B'$ meet

* If through a point O any chord be drawn to cut a circle in P and Q rectangle $OP \cdot OQ$ is called the *power* of the point with regard to the ci STEINER, *Crelle's Journal*, vol 1 (Berlin, 1826), Collected Works, vol 1 p We may then say that OC^2 is the *power* of the point O with regard to the c in Fig 42

† PONCELET, *Propr proj* Art 79

again in H Join GH *, and produce it to cut the axis in O. from the first circle

$$OA \cdot OB = OG \cdot OH \text{ (Euc. III. 36)},$$

and from the second

$$OA' \cdot OB' = OG \cdot OH,$$
$$OA \cdot OB = OA' \cdot OB'$$

O is therefore the middle point of the segment required points C and D will be the intersections with the axis of a circle described from the centre O with radius equal to the length of the tangent from O to either of the circles GAB, $G'A'B'$

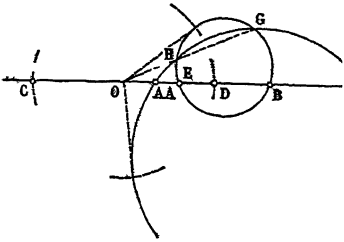


Fig 44

The problem admits of a real solution when the point O falls outside both the segments AB, $A'B'$, and consequently outside both the circles $GA'B'$ (Figs 43, 44) There is no real solution when the seg AB, $A'B'$ overlap (Fig 45), in this case O lies within both seg

71 Let $ABCD$ be a harmonic range, and let A and B (a pair of conjugates) approach indefinitely near to one another and ultimately coincide If C lie at an infinite distance, then D must coincide with A and B, since it must lie midway between these two points (Art 59) If C lie at a finite distance, and assume any position not coinciding with th or B, then equation (2) of Art 69 gives $CD = CA = CB$, i e D co with A and B

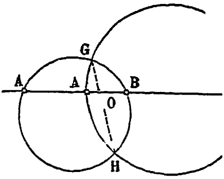


Fig 45

Again, let A and C (two non conjugate points) coincide, (the conjugate of A) lie at an infinite distance In this case lie midway between C and D, so that D will coincide with A If B lie at a finite distance, and assume any position not coin with that of A or C, then equation (1) of Art 69 gives $AD =$ the point D coincides with A and C So that '

If, of four points forming a harmonic range, any two coinci of the other two points will also coincide with them, and the is indeterminate

72 The theorem of Art 45 leads to the following result four elements A, B, C, D of a one dimensional geometric for

* GH is the radical axis of the two circles, and all points on it are power with regard to the circles

anharmonic ratios $(ABCD)$, $(BADC)$, $(CDAB)$, $(DCBA)$ are all equal to one another

I Four elements of such a form can be permuted in twenty-four different ways, so as to form the twenty-four different groups

$$
\begin{array}{cccc}
ABCD, & BADC, & CDAB, & DCBA, \\
ABDC, & BACD, & DCAB, & CDBA, \\
ACBD, & CADB, & BDAC, & DBCA, \\
ACDB, & CABD, & DBAC, & BDCA, \\
ADBC, & DACB, & BCAD, & CBDA, \\
ADCB, & DABC, & CBAD, & BCDA,
\end{array}
$$

here arranged in six lines of four each The four groups in each line are projective with one another (Art 45), and have therefore the same anharmonic ratio In order to determine the anharmonic ratios of all the twenty-four groups, it is only necessary to consider *one* group in each line, for example, the six groups in the first column These six groups are so related to each other that when any one of them is known the other five can be at once determined

II Consider the two groups $ABCD$ and $ABDC$, which are derived one from the other by interchanging the last two elements Their anharmonic ratios

$$(ABCD) \quad \text{or} \quad \frac{AC}{BC} \cdot \frac{AD}{BD}$$

and

$$(ABDC) \quad \text{or} \quad \frac{AD}{BD} \cdot \frac{AC}{BC}$$

are one the reciprocal of the other, thus

$$(ABCD)(ABDC) = 1 \qquad (1)$$

Similarly,

$$(ACBD)(ACDB) = 1, \qquad (2)$$

and

$$(ADBC)(ADCB) = 1 \qquad (3)$$

III Now if A, B, C, D are four collinear points, it has been seen (Art 62) that the identical relation

$$BC \cdot AD + CA \cdot BD + AB \cdot CD = 0$$

always holds Dividing by $BC \cdot AD$, we have

$$\frac{AC \cdot BD}{BC \cdot AD} + \frac{AB \cdot CD}{CB \cdot AD} = 1,$$

or

$$\frac{AC}{BC} \cdot \frac{AD}{BD} + \frac{AB}{CB} \cdot \frac{AD}{CD} = 1,$$

that is (Arts 63, 67),

$$(ABCD) + (ACBD) = 1 \qquad (4)$$

Similarly,

$$(ABDC) + (ADBC) = 1, \qquad (5)$$

and

$$(ACDB) + (ADCB) = 1 \qquad (6)$$

IV If λ denote the anharmonic ratio of the group $ABCD$, i.e.

$$(ABCD) = \lambda,$$

the formula (1) gives $\quad (ABDC) = \dfrac{1}{\lambda},$

and (4) gives $\quad (ACBD) = 1 - \lambda$

then by (2) $\quad (ACDB) = \dfrac{1}{1-\lambda},$

and by (6) $\quad (ADCB) = 1 - \dfrac{1}{1-\lambda} = \dfrac{\lambda}{\lambda - 1},$

and finally, by (3) or (5)

$$(ADBC) = \frac{\lambda - 1}{\lambda} \; *$$

V The six anharmonic ratios may also be expressed in ter[ms of] the angle of intersection θ of the circles described on the segm[ents] AB, CD as diameters, it being supposed that A and B are sepa[rated] by C and D It will be found that

$$(ABCD) = -\tan^2\frac{\theta}{2}, \qquad (ABDC) = -\cot^2\frac{\theta}{2},$$

$$(ACBD) = \sec^2\frac{\theta}{2}, \qquad (ACDB) = \cos^2\frac{\theta}{2},$$

$$(ADCB) = \sin^2\frac{\theta}{2}, \qquad (ADBC) = \operatorname{cosec}^2\frac{\theta}{2} \; \dagger$$

VI If in the group $ABCD$ two points A and B coincide, $AC = BC$, $AD = BD$, and therefore

$$(ABCD) = (AACD) = 1$$

But if $\lambda = 1$, the other anharmonic ratios become

$$(ACAD) = 1 - 1 = 0, \text{ and } (ACDA) = \infty,$$

thus when of four elements two coincide, the anharmonic ratios the values 1, 0, ∞

If $(ABCD) = -1$, i e if the range $ABCD$ is harmonic, the for[mulae] of (IV) give

$$(ACBD) = 2 \text{ and } (ACDB) = \tfrac{1}{2},$$

so that when the anharmonic ratio of four points has the value $\frac{1}{2}$, these points, taken in another order, form a harmonic range

VII Conversely, the anharmonic ratio of a range $ABCD$, no[ne of] whose points lies at infinity, cannot have any of the values 0, without some two of its points coinciding

For if in (IV) $\lambda = 0$, $\dfrac{AC}{BC} \dfrac{AD}{BD} = 0$, and either AC or BD vanish, i e either A coincides with C, or B with D

* Mobius, *loc cit*, p 249

† Casey, *On Cyclides and Sphero quartics* (Phil Trans 1871), p 70

If $\lambda = 1$, $(ACBD) = 1 - \lambda = 0$, so that either A coincides with B, or C with D

And if $\lambda = \infty$, $(ABDC) = \dfrac{1}{\lambda} = 0$, so that either A coincides with D, or B with C

VIII By considering the expressions given for the six anharmonic ratios in (IV) it is clear that whatever be the relative positions of the points A, B, C, D, two of the ratios (and their two reciprocals) are always positive and a third (and its reciprocal) negative, and thus we see that the anharmonic ratios of four points no two of which coincide may have all values positive or negative except $+1$, 0, or ∞

73 From the theorems of Arts 63 and 66, which express the necessary and sufficient condition that two ranges, each consisting of four elements, should be projectively related, we conclude that

If two geometric forms of one dimension are projective, then any two corresponding groups of four elements are equianharmonic *

As a particular case, to any four harmonic elements of the one form correspond four harmonic elements of the other (Art 51)

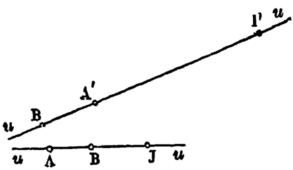


Fig 46

74 Let A, A' and B, B' be any two pairs of corresponding points of two projective ranges (Fig 46), let I be the point at infinity belonging to the first range, and I' the point corresponding to it in the second range, similarly let J' be the point at infinity belonging to the second range, and J its correspondent in the first range By Art 73

$$(ABIJ) = (A'B'I'J'),$$
$$(BAJI) = (A'B'I'J') \text{ (Art 72)},$$

from which, since I and J' lie at infinity,

$$BJ\ \ AJ = A'I'\ \ B'I' \text{ (Art 64)},$$

and
$$JA\ \ I'A' = JB\ \ I'B',$$

i e the product $JA\ I'A'$ has a constant value for all pairs of corresponding points †

[This proposition has already been proved in Art 63 for the particular case of two ranges in perspective]

* STEINER, *Systematische Entwickelung* (Berlin, 1832), p 33, § 10, Collected Works, ed Weierstrass (Berlin 1881), vol 1 p 262

† STEINER, *loc cit*, p 40 § 12, Collected Works, vol 1 p 267

75 In two homological figures, four collinear poi[nts or] four concurrent straight lines of the one figure form a [range] which is equianharmonic with that consisting of the po[ints or] lines corresponding to them in the other figure (Art. 73) [Let] O be the centre of homology, M and M' any pair of [corre]sponding points in the two figures, N and N' another [pair of] corresponding points lying on the ray OMM', and [X the] point in which this ray meets the axis of homology [Then, since] the points $OMNX$, $OM'N'X$ correspond severally t[o one] another,

$$(OXMN) = (OXM'N'),$$

or

$$\frac{OM}{MX} \frac{ON}{NX} = \frac{OM'}{M'X} \frac{ON'}{N'X},$$

$$\frac{OM}{MX} \frac{OM'}{M'X} = \frac{ON}{NX} \frac{ON'}{N'X},$$

and consequently the anharmonic ratio $(OXMM')$ is co[nstant] for all pairs of corresponding points M and M' taken on [the ray] OX passing through the centre of homology

Next let L and L' be another pair of corresponding [points] and Y the point in which the ray OLL' cuts the a[xis of] homology Since the straight lines LM, $L'M'$ must m[eet in] some point Z of the axis XY, it follows that $OYLL'$ is [a pro]jection of $OXMM'$ from Z as centre, and therefore

$$(OYLL') = (OXMM'),$$

consequently the anharmonic ratio $(OXMM')$ is const[ant for] all pairs of corresponding points in the plane

Consider now a pair of corresponding straight lines [a and] a', the axis of homology s, and the ray o joining the cen[tre of] homology O to the point aa' The pencil $osaa'$ is [cut by] every straight line through O in a range of four [points] analogous to $OXMM'$, consequently the anharmonic [ratio] $(osaa')$ is constant for all pairs of corresponding st[raight] lines a and a', and is equal to the anharmonic [ratio] $(OXMM')$

This anharmonic ratio is called the *coefficient* or *pa[rameter]* of the homology It is clear that two figures in hom[ology] can be constructed when, in addition to the centre an[d axis,] we are given the parameter of the homology

76 When the parameter of the homology is equal t[o −1,] all ranges and pencils similar to $OXMM'$, $osaa'$, are har[monic]

In this case the homology is called *harmonic* * or *involutorial*, and two corresponding points (or lines) correspond to one another doubly, that is to say, every point (or line) has the same correspondent whether it be regarded as belonging to the first or the second figure (See below, Arts 122, 123)

Harmonic homology presents two cases which deserve special notice (1) when the centre of homology is at an infinite distance, in the direction perpendicular to the axis of homology, (2) when the axis of homology is at an infinite distance In the first case we have what is called *symmetry with respect to an axis*, the axis of homology (in this case called also the *axis of symmetry*) bisects orthogonally the straight line joining any pair of corresponding points, and bisects also the angle included by any pair of corresponding straight lines The second case is called *symmetry with respect to a centre* The centre of homology (in this case called also the *centre of symmetry*) bisects the distance between any pair of corresponding points, and two corresponding straight lines are always parallel In each of these two cases the two figures are equal and similar (congruent) †, oppositely equal in the first case, and directly equal in the second

77 Considering again the general case of two homological figures, let a, b, m, n be four rays of a pencil in the first figure, and a', b', m', n' the straight lines corresponding to them in the second Then

$$(mnab) = (m'n'a'b')$$

Now let an arbitrary transversal be drawn to cut $mnab$ in $MNAB$, and draw the corresponding (or another) transversal to cut $m'n'a'b'$ in $M'N'A'B'$, then

$$(MNAB) = (M'N'A'B'),$$

or
$$\frac{MA}{MB} \quad \frac{M'A'}{M'B'} = \frac{NA}{NB} \quad \frac{N'A'}{N'B'}$$

Consequently, the ratio $\frac{MA}{MB} \quad \frac{M'A'}{M'B'}$ depends only on the straight lines ab (and $a'b'$), and not at all on the straight line m (or m')

The ratio $MA \quad NA$ is equal to that of the distances of the points M, N from the straight line a, which distances we may denote by (M, a), (N, a), thus

* BELLAVITIS, *Saggio di Geometria derivata* (*Nuovi Saggi* of the Academy of Padua vol IV 1838), § 50

† Two figures are said to be *congruent* when the one may be superposed upon the other so as exactly to coincide with it

$$\frac{(M,a)}{(M,b)} \quad \frac{(M',a')}{(M',b')} = \text{constant},$$

that is to say *

In two homological figures (or, more generally, in two projecti related figures) the ratio of the distances of a variable point M f two fixed straight lines a , b in the first figure bears a constant r to the analogous ratio of the distances of the corresponding p M′ from the corresponding straight lines a′ , b′ in the other figu

Suppose b to pass through the centre of homology O, t M and M' are collinear with O and b' coincades with b, so t

$$(M,b) \quad (M',b') = OM \quad OM',$$

and therefore

$$\frac{OM}{OM'} \quad \frac{(M,a)}{(M',a')} = \text{constant}$$

If N and N' are another pair of corresponding points have then

$$\frac{OM}{OM'} \quad \frac{(M,a)}{(M',a')} = \frac{ON}{ON'} \quad \frac{(N,a)}{(N',a')}$$

Now suppose the straight line a' to move away indefinit then a becomes the vanishing line in the first figure, the r $\frac{(M',a')}{(N',a')}$ will in the limit become equal to unity, and thus

$$\frac{OM}{OM'} \quad (M,a) = \frac{ON}{ON'} \quad (N,a)$$
$$= \text{constant},$$

in other words †

In two homological figures, the ratio of the distances of any p in the first figure from the centre of homology and from the vanis line respectively, varies directly as the distance of the correspo point in the second figure from the centre of homology

* CHASLES, *Geometrie supérieure*, Art 51

† CHASLES, *Sections coniques*, Art 267

CHAPTER X

CONSTRUCTION OF PROJECTIVE FORMS

78 LET ABC and $A'B'C'$ be two triads of correspondin elements of two projective forms of one dimension (Fig 47 and imagine any series of operations (of projection and sectio by which we may hav passed from ABC t $A'B'C'$ Then whateve this series be*, it wi also lead from any othe element D of the fir form to the element I which corresponds to in the second For if could give, as the resu of these operations, a element D'' different from D', then the anharmonic ratic $(ABCD)$ and $(A'B'C'D'')$ would be equal, but by hypothes $(ABCD) = (A'B'C'D')$, therefore $(A'B'C'D') = (A'B'C'D''$ which is impossible unless D'' coincide with D' (Art 65)

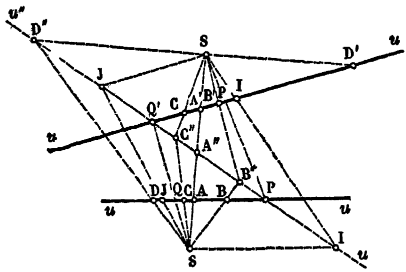


Fig 47

79 THEOREM (converse to that of Art 73)

Given two forms of one dimension, if to the elements A, B, C, D, of the one correspond respectively the elements A', B', C', D', of the other in such a manner that any four elements of the first for are equianharmonic with the four corresponding elements of the secon then the two forms are projective

For every series of operations (of projection or section which leads from the triad ABC to the triad $A'B'C'$, leads the same time from the element D to another element D'' suc that $(ABCD) = (A'B'C'D'')$ But $(ABCD) = (A'B'C'D')$ b hypothesis, therefore $(A'B'C'D') = (A'B'C'D'')$, and D'' mu coincide with D' (Art 65) And since the same conclusion

* In Fig 47 the series of operations is a projection from S, a section by u a projection from S and a section by u

true for any other pair whatever of corresponding el
follows that the two forms are projective (Art. 40).

80 From Art 78 the following may be deduced
ticular case

If among the elements of two projective forms of one there are two corresponding triads ABC and A′B′C′ wh perspective, then the two forms themselves are in perspectiv

(1) If, for example, the forms are two ranges ABC
$A'B'C'D'$, then if the three straight lines AA',
meet in a point S, the other analogous lines DD',
pass through S (Figs 19, 40).

Suppose, as a particular case, that the points A, A
(Fig 22), so that the two ranges have a pair of corre
points A and A' united in the point of intersection
bases* The triads ABC, $A'B'C'$ are in perspective, th
of perspective being the point where BB' and C
accordingly

If two projective ranges have a self-corresponding point, perspective

Conversely it is evident that *two ranges* *which* are in *i.e.* *spective have always a self-corresponding point*

(2) Again, if the two forms are two flat pencils a
$a'b'c'd'$ lying in the same plane, then if the th
aa', bb', cc' lie on one straight line s, the analogous p
will all lie on the same straight line (Fig 20) If
lie altogether at infinity, we have the following prop

If, in two projective flat pencils, three pairs of corresponding rays are parallel to one another, then every pair of corresponding rays are parallel to one another

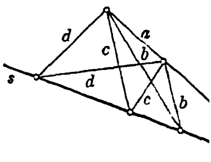


Fig 48

The hypothesis is satisfied in the particular case where the rays a and a' coincide (Fig 48), so that the two pencils have a self corresponding straight line which joins their centres, then s is th line joining bb' and cc' Accordingly

* In the case of two projective forms we shall in future emp *self corresponding* to denote an element which is such that it coin correspondent, thus A or A' above may be called a *self correspondin* two ranges

When two projective flat pencils (lying in the same plane) have a [illegible] *ray, they are in perspective*

Conversely, *two coplanar flat pencils which are in perspective have always a self-corresponding ray*

(3) If one of the systems is a range $ABCD$ and the other a flat pencil $abcd$ (Fig 28), the hypothesis amounts to assuming that the rays a, b, c pass respectively through the points A, B, C, then we conclude that also d, will pass through D, &c

81 Two ranges may be *superposed* one upon the other, so as to lie upon the same straight line or base, in which case they may be said to be *collinear* For example, if two pencils (in the same plane) $S \equiv abc$ and $O \equiv a'b'c'$ (Fig 49) are cut by the same transversal, they will determine upon it two ranges ABC, $A'B'C'$ which will be projectively related if the two pencils are so The question arises whether there exist in this case any self-corresponding points, *i e* whether two corresponding points of the two ranges coincide in any point of the transversal

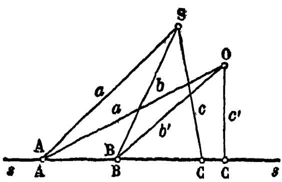


Fig 49

If, for instance, the transversal s be drawn so as to pass through the points aa' and bb', then A will coincide with A', and B with B', in this case consequently there are *two* self-corresponding points

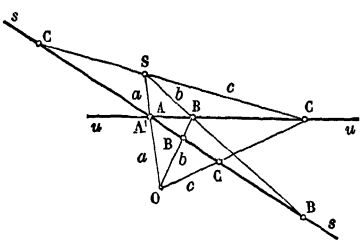


Fig 50

Again, if a range u be projected (Fig 50) from two centres S and O (lying in the same plane with u), two flat pencils abc and $a'b'c'$ will be formed, which have a pair of corresponding rays a, a' united in the line SO And if a transversal s be drawn through the point in which this line cuts u, we shall obtain two projective ranges ABC, $A'B'C'$ lying on a common base s, and such that they have *one* self-corresponding point AA'

And lastly we shall see hereafter (Art 109) that it is possible

for two collinear projective ranges to be such as to h
self-corresponding point.

So also two flat pencils (in the same plane) may l
common centre, in which case they may be termed *conc*
such pencils are formed when two different ranges ar
jected from the same centre (Fig 51). And two axial
may have a common axis, such pencils are formed when we project two different ranges from the same axis, or the same flat pencil from different centres Again, if two sheaves are cut by the same plane, two plane figures are obtained, if, on the other hand, two plane figures are projected from the same centre, two concentric sheaves are formed these cases the forms in question may be said to be *sup*
one upon the other, and the investigation of the
corresponding elements, when the two forms are proje
related, is of great importance The complete invest
will be given later on, in Chapter XVIII, at present
only prove the following Theorem

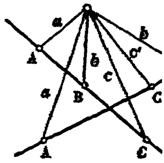


Fig 51

82 THEOREM *Two superposed projective (one-dime
forms either have* at most two *self-corresponding elements*,
every *element coincides with its correspondent*

For if there could be three self-corresponding el
A, B, C suppose, then if D and D' are any other pair
responding points, we should have (Art 73) $(ABCD) = (A$
and consequently (Art 65) D would coincide with D'
then the two forms are identical, they cannot have mo
two self-corresponding elements

83 THEOREM (converse to that of Art 53) *If a one
sional form consisting of four elements A, B, C, D is project
a second form deduced from it by interchanging two of the e
(e g $BACD$), then the form will be a harmonic one, and
interchanged elements will be conjugate to each other*

First Proof If $(ABCD) = (BACD)$, then (Art 72 IV)

$\lambda^2 = 1$, and since we cannot take $\lambda = +1$ (Art 72
we must have $\lambda = -1$, *i e* the form is a harmonic one

Second Proof Suppose, for example, that A, B, C, D a

collinear points (Fig 52) Let K, M, Q, D be a projection these points on any straight line through D, made from . arbitrary centre L Since $ABCD$ is projective with KMQ and also (by hyp) with $BACD$, the forms $KMQD$ and BAC are projective with one anoth And they have a self-corr sponding point D, consequent they are in perspective (Art 8 and KB, MA, QC will meet one point N But this bei the case, we have a comple

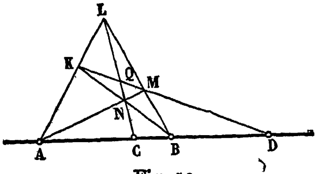


Fig 52

quadrangle $KLMN$, of which one pair of opposite sides me in A, another such pair in B, while the fifth and sixth sic pass respectively through C and D Accordingly (Art 4 $ABCD$ is a harmonic range

84 Let there be given two projectively related geomet forms of one dimension Any series of operations which su fices to derive three elements of the one from the three cor sponding elements of the other will enable us to pass fr the one form to the other (Art 78), and any two given tria of elements are always projective, *i e* can be derived one fr the other by means of a certain number of projections a sections Hence we conclude that

Given three pairs of corresponding elements of two projective for of one dimension, any number of other pairs of corresponding eleme can be constructed

We proceed to illustrate this by two examples, taki (1) two ranges and (2) two flat pencils, the forms bei in each case supposed to lie in one plane

Given (Fig 53) *three pairs of corresponding points A and A', B and B', C and C', of the projective ranges u and u', to construct these ranges*

We proceed as in Art 44 On the straight line which joins any two of the corresponding points, say A and A', take two arbitrary points S and S' Join SB, $S'B'$ cutting one another in B'', and SC, $S'C'$ cutting one another in

Given (Fig 54) *three pa of corresponding rays a and b and b', c and c', of the pro tive pencils U and U', to c struct these pencils*

Through the point of in section of any two of the c responding rays, say a and draw two arbitrary transver s and s' Join the points sb $s'b'$ by the straight line b'', the points sc and $s'c'$ by

C'', join $B''C''$, and let it cut AA' in A'' The operations which enable us to pass from ABC to

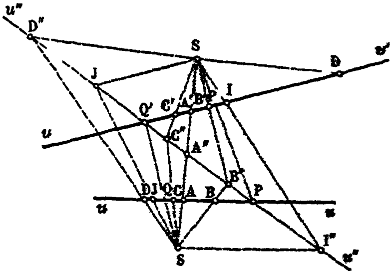


Fig 53

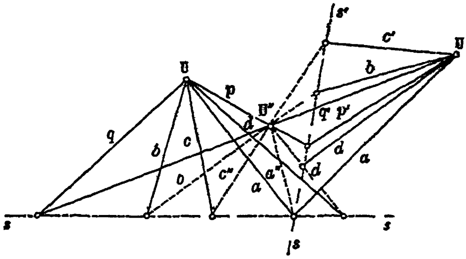


Fig 54

$A'B'C'$ are 1 a projection from S, 2 a section by u'' (the line on which lie the points A'', B'', C''), 3 a projection from S', 4 a section by u' The same operations lead from any other given point D on u to the corresponding point D' on u', so that the rays SD and $S'D'$ must intersect in a point D'' of the fixed straight line u''

In this manner a range
$$u'' \equiv A''B''C''D''$$
is obtained which is in perspective both with u and with u'

straight line c'', and let a straight line joining th $b''c''$ and aa' The op which enable us to pass to $a'b'c'$ are 1 a secti 2 a projection from t U'' where a'', b'', c'' m section by s', 4 a p from U' The same o lead from any other giv of the pencil U to the cor ing ray d' of the pencil U the points sd and $s'd'$ mu a straight line d'' whic through the fixed point

In this manner a penc
$$U'' \equiv a''b''c''d''$$
is obtained which is in tive both with U and wit

In the preceding construction (left), D is any arbitrary po If D be taken to be the point at infinity on u, then (Fig will be parallel to v, in order therefore to find the poi

which corresponds to the point at infinity on u, draw SI'' parallel u to cut u'' in I'', then join $S'I''$, which will cut u' in the requi point I' Similarly, if the ray through S' parallel to u' cuts u'' J'', and SJ'' be joined, this will cut u in J, the point on u wh corresponds to the point at infinity on u'

If D be taken at P, the point where u and u'' meet, then D'' also coincides with P, and the point P' on u' corresponding to the point P on u is found as the intersection of $S'P$ with u'

Similarly, if Q' be the point of intersection of u' and u'', the point on u corresponding to Q' on u' is Q, where SQ' cuts u

85 The only condition to which the centres S and S' are subject is that they are to lie upon the straight line which joins a pair of corresponding points, in other respects their position is arbitrary We may then for instance take S at A' and S' at A (Fig 55) Then the ray $S'P$ coincides with u, and P' is accordingly the point of intersection of u and u' So too the ray SQ' coincides with u, and Q also lies at the point uu'

If then we take the points A' and A as the centres S and S respectively, the straight line u'' will cut the bases u and u' respectively in P and Q', the points which correspond to the point uu' regarded in the first instance as the point P' of the line u' and in the second instance as the point Q of the line u

Now in the construction of the preceding Art, the straight line u'' was found at the locus of

In the preceding construct (right), d is any arbitrary passing through U If it be tak to be p, the line joining U to l then the corresponding ray p' the pencil U' is the line join the point U' to the point $s'p$

Similarly, if q' be the $U'U''$ of the pencil U', the ra corresponding to it in the per U is that which joins the poi U and sq'

The only condition to wh the transversals s and s' are su ject is that they are to p through the point of intersecti of a pair of corresponding ray in other respects their position arbitrary We may then for stance take a' for s and a fo (Fig 56) Then the point coincides with U, and p' is cordingly the straight line U So too the point sq' coinci with U', and q also must be straight line UU'

If then we take the rays and a as the transversals s a s' respectively, the point U'' w be the intersection of the rays and q' which correspond to t straight line UU', regarded the first instance as the ray p' the pencil U', and in the seco instance as the ray q of t pencil U

Now in the construction of t preceding Art, the point U'' w found as the centre of perspect

the points of intersection of pairs of corresponding rays of the pencils in perspective
$S(ABCD\ \ )$ and $S'(A'B'C'D'\ \ )$

The straight line u'' obtained by the construction of the present Art. is in like manner the locus of the points of intersection of pairs of corresponding rays of the pencils $A'(ABCD\ \ )$ and $A(A'B'C'D'\ \ )$, *i e* the locus of the points in which the pairs of lines $A'B$ and AB', $A'C$ and AC', $A'D$ and AD', intersect.

of the ranges in perspective
$s(abcd\ \ )$ and $s'(a'b'c'd'$

The point U'' obtained b construction of the presen is in like manner the cen perspective of the ranges
$a'(abcd\ \ )$ and $a(a'b'c'd'$
i e. the point in which the joining the pairs of correspo points $a'b$ and ab', $a'c$ ar $a'd$ and ad', meet.

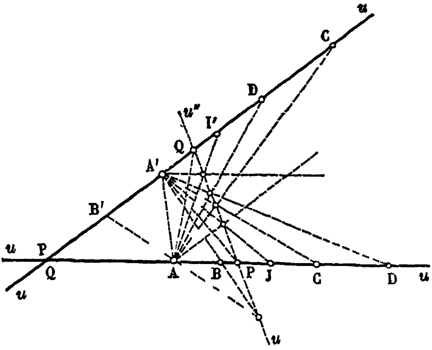


Fig 55

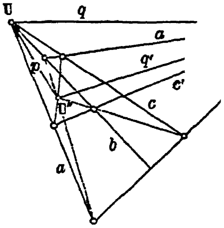


Fig 56

If in place of A' and A any other pair of points B' and B, or C' and C, be taken as centres of the auxiliary pencils S and S', the straight line u'' must still cut the two bases u and u' in the points P and Q', *i e* the straight line u'' remains the same

If then $ABC \quad MN$ and $A'B'C' \quad M'N'$ are two projective ranges (in the same plane), every pair of straight lines such as MN' and $M'N$ intersect in points lying on a fixed straight line This straight line passes through those points which cor-

If in place of a' and other pair of rays b' and and c, be taken as trans the point U'' must still intersection of p and q', point U'' remains the sam

If then $abc \quad mn$ $a'b'c' \quad m'n'$ are two tive pencils (in the same every straight line which pair of points such as m $m'n$ passes through a fixec This point is the intersec those rays which corresp

respond in each range to the point of intersection of their bases when regarded as a point of the other range

86 If the two ranges u and u' are in perspective (Fig 57) the points P and Q' will coincide with the point O in which the bases u and u' meet, and since the straight line which is the locus of the points $(AB', A'B)$, $(AC', A'C)$, $(AD', A'D)$, and the straight line which is the locus of the points $(BA', B'A)$, $(BC', B'C)$, $(BD', B'D)$, have two points in common, viz O and $(AB', A'B)$, these straight lines must coincide altogether This being so, $AA'BB'$ is a complete quadrangle, whose diagonal points are O, S (the point where AA', BB', meet), and M (the point of intersection of AB' and $A'B$), consequently (Art 57) the straight lines u and u' are harmonic conjugates with regard to the straight lines u'' and OS If therefore two transversals u and u' cut a flat pencil $(a, b, c, \ )$ in the points (A, A'), (B, B'), (C, C') , then the points of intersection of the pairs of straight lines AB' and $A\ B$, AC' and $A'C$, BC' and $B'C$, lie on one and the same straight line u'', which passes through the point uu', and the straight line joining uu' to the centre of the pencil is the harmonic conjugate of u'' with respect to u and u'

From this follows the solution of the problem

To draw the straight line connecting a given point M with the

each pencil to the straight l joining the centres of the pen when regarded as a ray of other pencil

If the two pencils U and are in perspective (Fig 59) rays p and q' will coincide w the straight line UU', and si through the point of intersect of the rays $(ab', a'b)$, $(ac', a$ $(ad', a'd)$, and through point of intersection of the r $(ba', b'a)$, $(bc', b'c)$, $(bd', b'd)$ pass two different straight li viz UU' and $(ab', a'b)$, th points must coincide This be so, $aa'bb'$ is a complete qua lateral, whose diagonals are U s (the straight line on wh aa', bb', intersect), and m straight line which joins ab' $a'b$), consequently (Art 56) points U and U' are harm conjugates with regard to U'' the point in which s meets U If therefore a range be proje from two points U and U' by rays (a, a'), (b, b'), (c, c') , t the straight lines which join pairs of points $(ab', a'b)$, $(ac'$, $(bc', b'c)$, meet in one and same point U'', which lies on line UU', and the point w the straight line UU' cuts the of the range is the harmonic jugate of U'' with respect t and U'

From this follows the solu of the problem

To construct the point whe given straight line m would b

inaccessible point of intersection of two given straight lines u and u'

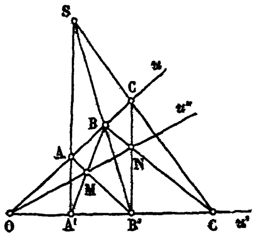


Fig 57

Through M (Figs. 57 and 58) draw two straight lines to cut u in A and B, and u' in B' and A'

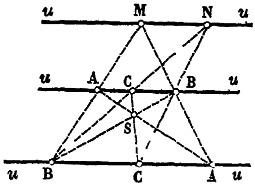


Fig 58

respectively, join AA', BB' meeting in S Through S draw any straight line to cut u in C and u' in C', and join BC', $B'C$, intersecting in N The straight line joining M and N will be the line u'' required

tersected by a straight line [
which cannot be drawn, but
is determined by its p
through two given points l
U'

On m (Fig 59) take two]
and join them to U b
straight lines a and b, and

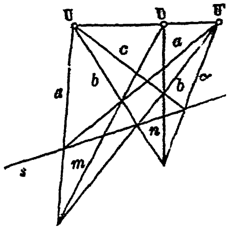


Fig 59

by the straight lines b' a
let s be the straight line]
the points of intersection o
and b, b' On s take any
point and join it to U, U' l
straight lines c, c' respec
The straight line n which
the points bc' and $b'c$ will
in the point U'' required

If the straight lines u and u' are parallel to one another (Fi
the preceding construction gives the solution of the problem
two parallel straight lines, to draw through a given point a st
line parallel to them, making use of the ruler only

87 If in the theorem of the preceding article the flat pencil

If in the theorem of th
ceding article the range (

consist of only three rays, the theorem may be enunciated as follows, with reference to the three pairs of points AA', BB', CC'

If a hexagon (six-point) $AB'CA'BC'$ (Fig 60) has its vertices of odd order (1st, 3rd, and 5th)

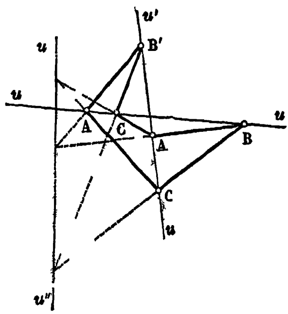


Fig 60

on one straight line u, and its vertices of even order (2nd, 4th, and 6th) on another straight line u', then the three pairs of opposite sides (AB' and AB', $B'C$ and BC', CA' and $C'A$) meet in three points lying on one and the same straight line u'' *

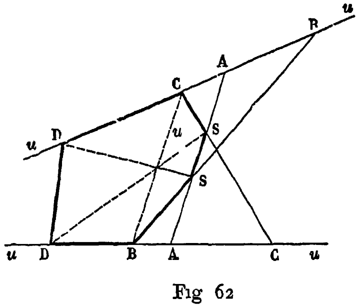


Fig 62

88 Returning to the construction of Art 84 (left), let the

of only three points, the theorem may be enunciated as follows with reference to the three pairs of rays aa', bb', cc'

If a hexagon (six-side) $ab'ca'bc'$ (Fig 61) be such that its sides of odd order (1st, 3rd, and 5th)

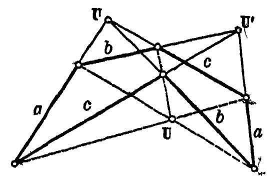


Fig 61

meet in one point U, and its sides of even order (2nd, 4th, and 6th) meet in another point U', then the three straight lines which join the pairs of opposite vertices (ab' and $a'b$, $b'c$ and bc', ca' and $c'a$) pass through one and the same point U''

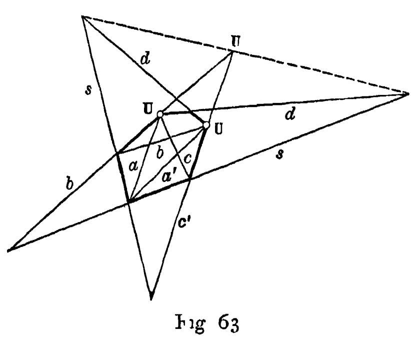


Fig 63

Returning to the construction of Art 84 (right), let the straight

* PAPPUS, *loc cit*, Book vii prop 139

centre S be taken at the point where AA' meets BB', and the centre S' at the point where AA' meets CC' (Fig 62) Then since SB, $S'B'$ meet in B', and SC, $S'C'$ in C, therefore $B'C$ is the straight line u'' Consequently any other pair of corresponding points D and D' are constructed by observing that the straight lines SD, $S'D'$ must meet on $B'C$

line joining the points aa', taken as the transversal s, that joining the points a as the transversal s' (Fig Then since the line joining points sb, $s'b'$ is b, and the joining the points sc, $s'c'$ therefore bc' is the point Consequently any other pa corresponding rays d and d constructed by observing tha points sd, $s'd'$ must be coll with bc'

From a consideration of the figure $SS'CDD'B$, which is a hexagon, we derive the theorem

In a hexagon, of which two sides are segments of the bases of two projective ranges, and the four others are the straight lines connecting four pairs of corresponding points, the straight lines which join the three pairs of opposite vertices are concurrent

From a consideration of figure $ss'cdd'b$, which is a gon (six-side) we derive theorem

In a hexagon, of which two tices are the centres of two jective pencils, and the four are the points of intersectu four pairs of corresponding the three points in which the of opposite sides meet one a are collinear

89 If in the problem of Art 84 (left) the three straight lines AA', BB', CC' passed through the same point S (if, for example, A and A' coincided), then the two ranges would be in perspective, we should therefore only have to draw rays through S in order to obtain any number of pairs of corresponding points (Fig 19)

If the three points aa', in Art 84 (right) lay o same straight line s (if, fo ample, a and a' coincided), the two pencils would be in spective, we should the only have to connect the centres of the pencils with point of s in order to obtai number of pairs of correspo rays (Fig 20)

90 If the two ranges u and u' (Art 84, left) are superpose upon the other, i e if the six given points $AA'BB'CC'$ lie o same straight line (Fig 64), we first project u' from an arbi centre S' on an arbitrary straight line u_1, and then proceed to the construction for the case of the ranges $u \equiv (ABC \quad)$ $u_1 \equiv (A_1B_1C_1 \quad)$, i e to construct with regard to the pairs of p (AA_1), (BB_1), (CC_1) in the way shown in Art 84 A pair of c sponding points D and D_1 of the ranges u and u_1 having been f

the ray $S'D_1$ determines upon u' the point D' which corresponds to D

The construction is simpler in the case where two corresponding points A and A' coincide (Fig 65) When this is so, if u_1 be drawn through A, the range u_1 will be in perspective with u, thus, after having projected u' upon u_1 from an arbitrary centre S', if S be the point where BB' and CC_1' meet, it is only necessary further to project u from S upon u_1, and then u_1 from S' upon u'

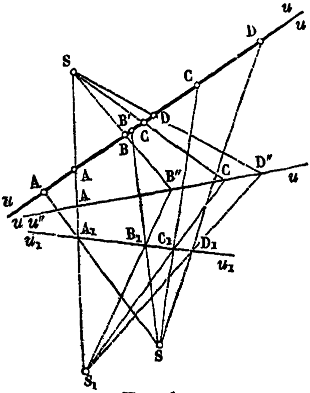


Fig 64

The two collinear ranges u and u' have in general two self corresponding points, one at AA', and a second at the point of intersection of their common base line with the straight line SS'

If then SS' passes through the point uu_1, the two ranges u and u' have only *one* self-corresponding point If it were desired to construct upon a given straight line two collinear ranges having A and A' for a pair of corresponding points, and a single self-corresponding point at M (Fig 66), we should proceed as follows Take

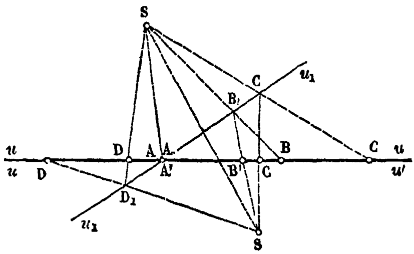


Fig 65

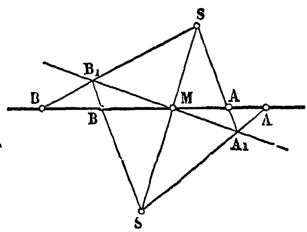


Fig 66

any point S', and draw any straight line u_1 through M, project A' from S' on u_1, join the point A_1 so found to A, and let AA_1 meet $S'M$ in S Then to find the point on u' which corresponds to any point B on u, project B from S into B_1, and then B_1 from S' into B', this last is the point required

If the two pencils U, U' (Art 84, right) are concentric, *i e* if the six rays $aa'bb'cc'$ pass all through one point, we first cut $a'b'c'$ by a transversal and then project the points of intersection from an arbitrary centre U_1 If $a_1b_1c_1$ are the projecting rays, we have then

only to consider the non-concentric pencils U and $U_1 \equiv (a_1 b_1 c_1)$ we may cut abc by a transversal in the points ABC, and $a'b'$ another transversal in $A'B'C'$, and then proceed with the two r ABC , $A'B'C'$ in the manner explained above.

The figures corresponding to these constructions are not g the student is left to draw them for himself. He will see th these cases also the constructions admit of considerable simplifi if, among the given rays, there be one which is self-correspondin for example, a and a' coalesce and form a single ray, &c.

91 Consider two projective (homographic) plane figures π', as has already been seen (Art. 40), any two corresponding st lines are the bases of two projective ranges, and any two corres ing points are the centres of two projective pencils.

If the two figures have three self-corresponding points lying straight line, this straight line s will correspond to itself, for i contain two projective ranges which have three self-correspo points, and every point of the straight line s will therefore (Ar be a self-corresponding point. Consequently every pair of correspo straight lines of π and π' will meet in some point on s, and the the two figures are in perspective (or in homology in the case they are coplanar)

92 If two projective plane figures which are coplanar have self-corresponding rays all meeting in a point O, this point w the centre of two corresponding (and therefore projective) p which have three self-corresponding rays, therefore (Art 82) ray through O will be a self-corresponding one Hence it f that every pair of corresponding points will be collinear wi therefore the two figures are in homology

93 *If two projective plane figures which are coplanar hav self-corresponding points A, B, C, D, no three of which are coll then will every point coincide with its correspondent*

For the straight lines AB, AC, AD, BC, BD, CD are al corresponding, therefore the points of intersection of AB an AC and BD, BC and AD, *i e* the diagonal points of the quad $ABCD$, are all self corresponding Since the three points A, (AB) (CD) are self-corresponding, every point on the straigh AB coincides with its correspondent, and the same may be prove for the other five sides of the quadrangle If now a straight l drawn arbitrarily in the plane, there will be six points on it are self-corresponding, those namely in which it is cut by th sides of the quadrangle, and therefore every point on the st line is a self-corresponding one, which proves the proposition

In a similar manner it may be shown that *if two coplana jective figures have four self corresponding straight lines a, b,*

forming a complete quadrilateral (i e such that no three of them are concurrent), then every straight line will coincide with its correspondent

94. THEOREM *Two plane quadrangles $ABCD$, $A'B'C'D'$ are always projective*

(1) Suppose the two quadrangles to lie in different planes π, π' Join AA', and on it take an arbitrary point S (different from A'), and through A draw an arbitrary plane π'' (distinct from π), then from S as centre project A', B', C', D' upon π'' and let A'', B'', C'', D'' be their respective projections (A'' therefore coinciding with A)

In the plane π join AB, CD, and let them meet in E, so too in the plane π'' join $A''B''$, $C''D''$, and let these meet in E'' The straight lines ABE, $A''B''E''$ lie in one plane since they meet each other in the point $A \equiv A''$, therefore BB'' and EE'' will meet one another in some point S_1

Now let a new plane π''' (distinct from π) be drawn through the straight line ABE, and let the points A'', B'', C'', D'', E'' be projected from S_1 as centre upon π''' Let A''', B''', C''', D''', E''' be their respective projections, where A''', B''', E''' are collinear and coincide with A, B, E respectively, and C''', D''', E''' are collinear also, since their correspondents C'', D'', E'' are collinear The straight lines CDE, $C'''D'''E'''$ lie in one plane since they meet each other in the point $E \equiv E'''$, therefore CC''' and DD''' will meet one another in some point S_2 If now the points A''', B''', C''', D''' be projected from S_2 as centre upon the plane π, their projections will evidently be A, B, C, D

The quadrangle $ABCD$ may therefore be derived from the quadrangle $A'B'C'D'$ by first projecting the latter from S as centre upon the plane π'', then projecting the new quadrangle so formed in the plane π'' from S_1 upon π''', and lastly projecting the quadrangle so formed in the plane π''' from S_2 upon π, that is to say, by means of three projections and three sections *

(2) The case of two quadrangles lying in the same plane reduces to the preceding one, if we begin by projecting one of the quadrangles upon another plane

(3) If the two quadrangles (lying in different planes) have a pair of their vertices coincident, say D and D', then two projections will suffice to enable us to pass from the one to the other, or, what amounts to the same thing, a third quadrangle can be constructed which is in perspective with each of the given ones $ABCD$, $A'B'C'D'$

For let there be drawn through D two straight lines s and s', one in each of the planes, let s cut the sides of the triangle ABC in

* GRASSMANN *Die stereometrischen Gleichungen dritten Grades und die dadurch erzeugten Oberflachen* Crelle's Journal, vol 49 § 4 (Berlin, 1855)

L, M, N respectively, and let s' cut the sides of the triangle A' L', M', N' respectively Then in the plane ss' the straight lin MM', NN' will form a triangle which is in perspective at o ABC and with $A'B'C'$

(4) If the quadrangles (still supposed to lie in different have two pairs of their vertices $C \equiv C'$, $D \equiv D'$ coincident, the straight lines AA', BB' meet one another the quadrangle directly in perspective, the point of intersection O of AA' a being the centre of projection, so that we can pass at once f one quadrangle to the other by one projection from O If A are not in the same plane, so that they do not meet one anoth through CD let an arbitrary plane π'' be drawn, and in it straight line be drawn which meets AB and $A'B'$ If in this line two arbitrary points A'', B'' be taken, then $A''B''C''D'$ a quadrangle which is in perspective at once with $ABCD$ a $A'B'C'D'$

95 From the theorem just proved it follows that two p plane figures π and π' can be constructed when we are gi corresponding quadrangles $ABCD$, $A'B'C'D'$, for the o (projections and sections) which serve to derive $A'B'C'$ $ABCD$ will lead from any point or straight line whatever of corresponding point or straight line of π', and *vice versa*

Or, again, it may be supposed that two corresponding quad are given For if in these two corresponding pairs of opp tices be taken, we have thus two corresponding quadrangles, operations (projections and sections) which enable us t one of these quadrangles from the other will also derive quadrilateral from the other

96 Two plane figures may also be made projective in manner, leaving out of consideration the relative positio planes in which they lie, we may operate on each of th separately* Suppose that we are given, as correspondin another, two complete quadrilaterals $abcd$, $a'b'c'd'$ We constructing, on each pair of corresponding sides, such as the projective ranges which are determined by the three corresponding points ab and $a'b'$, ac and $a'c'$, ad and a' done, to every point of any of the four straight lines a b, correspond a determinate point of the corresponding lin other figure

(1) Now let in the first figure a transversal m be draw a, b, c, d in A, B, C, D respectively, then the points A' B which correspond to these in the second figure will in like m on a straight line m'

* Staudt *Geom der Lage*, Art 130

For, considering the triangle abc, cut by the transversals d anc the product of the three anharmonic ratios

$$a\,(bcdm)\,,\; b\,(cadm)\,,\; c\,(abdm)$$

is equal to $+1$ (Art 140), but these anharmonic ratios are e(respectively to the following

$$a'(b'c'd')\;\; A'\,,\; b'(c'a'd')\;\; B'\,,\; c'(a'b'd')\;\; C',$$

so that the product of these last three is also equal to $+1$ therefore, since the points $a'd'$, $b'd'$, $c'd'$ are collinear, the po A', B', C' are also collinear (Art 140)

By considering in the same manner the triangle abd, cut by transversals c and m, it can be shown that A', B', D' are collin it follows then that the four points A', B', C', D' all lie on same straight line m', the correspondent of m

This proof holds good also when m passes through one of vertices of the quadrilateral $abcd$, if for example m pass thro cd, the anharmonic ratios $c(abdm)$, $d(abcm)$ will each be equal to the reasoning, however, remains unaltered

Thus every pair of corresponding vertices of the quadrilate $abcd$, $a'b'c'd'$ (for example cd and $c'd'$) become the centres of projective pencils, in which to c, d, $(cd)(ab)$ correspond c', d', $(c'd')($ respectively, and to any ray cutting a, b in two points P, Q responds a ray cutting a', b' in the two corresponding points P', (

(2) The two ranges $ABCD$, $A'B'C'D'$ in which the sides of quadrilaterals $abcd$, $a'b'c'd'$ are respectively cut by two correspon straight lines m, m' are projective

For, considering the triangle bcm, cut by the transversals a an the product of the anharmonic ratios of the three ranges

$$\begin{array}{cccc} bc\,, & B\,, & ba\,, & bd \\ C\,, & cb\,, & ca\,, & cd \\ B\,, & C\,, & A\,, & D \end{array}$$

is equal to $+1$ And considering in like manner in the other p the triangle $b'c'm'$, cut by the transversals a' and d', the produ the anharmonic ratios of the three ranges

$$\begin{array}{cccc} b'c'\,, & B'\,, & b'a'\,, & b'd' \\ C'\,, & c'b'\,, & c'a'\,, & c'd' \\ B'\,, & C'\,, & A'\,, & D' \end{array}$$

is also equal to $+1$ But the range in which b is cut by the pe $cmad$ is equianharmonic with the range in which b' is cut by pencil $c'm'a'd'$, i e the ranges

$$\begin{array}{cccc} bc\,, & B\,, & ba\,, & bd \\ b'c'\,, & B'\,, & b'a'\,, & b'd' \end{array}$$

are equianharmonic, and for a similar reason the ranges

$$\begin{matrix} C\,, & cb\,, & ca\,, & cd \\ C'\,, & c'b'\,, & c'a'\,, & c'd' \end{matrix}$$

are equianharmonic Therefore the ranges

$$\begin{matrix} B\,, & C\,, & A\,, & D \\ B'\,, & C'\,, & A'\,, & D' \end{matrix}$$

will be equianharmonic and therefore projective, whence it that the projective ranges m and m' are determined by m the pairs of corresponding points lying on a and a', b c and c'

(3) If the straight line m turn round a fixed point M, i also will revolve round a fixed point.

For by hypothesis the points A and B, in which m cuts describe two ranges in perspective whose self-corresponding ab Similarly the points A', B' describe two ranges, whicl respectively projective with the ranges on a, b, are projecti one another, and which are further seen to be in pers since they have a self-corresponding point $a'b'$ Consequei straight line m' will always pass through a fixed point correspondent of M, and will therefore trace out a penc pencils generated by m and m' are projective, since th are projective in which they are cut by a pair of corres sides of the quadrilaterals, e g by a and a' To the rays pencil M which pass respectively through the vertices ab, bc, bd, cd of the quadrilateral $abcd$ correspond the rays of th M' which pass respectively through the vertices $a'b'$, $a'c'$, $b'd'$, $c'd'$ of the quadrilateral $a'b'c'd'$

This reasoning holds good also when the point M, roun m turns, lies upon one of the sides of the quadrilateral, example, because we still obtain two ranges in perspective u of the other sides Since c is now a ray of the pencil M, c the corresponding ray of the pencil M', that is to say, M' w c' If M be taken at one of the vertices, as cd, then M' will with $c'd'$, &c

(4) Now suppose the pencil M to be cut by a transversal n pencil M' to be cut by the corresponding straight line n' W point mn describes the range n, the corresponding point n describe the range n', and these two ranges will be projecti they are sections of two projective pencils When the point on one of the sides of the quadrilateral $abcd$, the point $m'n'$ on the corresponding side of the quadrilateral $a'b'c'd'$, there two projective ranges are the same as those which it has been shown may be obtained by starting from the pairs of cor ing points on a and a', b and b', c and c'

In this manner the two planes become related to one another in such a way that there corresponds uniquely to every point in the one a point in the other, to every straight line a straight line, to every range a projective range, to every pencil a projective pencil The two figures thus obtained are the same as those which can be obtained as explained above (Art 95) by means of successive projections and sections, so arranged as to lead from the quadrilateral *abcd* to the quadrilateral *a′b′c′d′*. For the two figures π' derived from π by means of these two processes have four self-corresponding straight lines *a′*, *b′*, *c′*, *d′* forming a quadrilateral, and therefore (Art 93) every element (point or straight line) of the one must coincide with the corresponding element in the other, *i e* the two figures must be identical

97 THEOREM *Any two projective plane figures (the straight line at infinity in which are not corresponding lines) can be superposed one upon the other so as to become homological*

Let *i*, *j′* be the vanishing lines of the two figures—*i e* the straight lines in each which correspond respectively to the straight line at infinity in the other In the first place let one of the figures be superposed upon the other in such a manner that *i* and *j′* may be parallel to one another Since to any point *M* on *i* corresponds a point at infinity in the second figure, to the pencil of straight lines in the first figure which meet in *M* corresponds in the second figure a pencil of parallel rays Through *M* draw the straight line *m* parallel to these rays, then *m* will be parallel to its correspondent *m′* Similarly let a second point *N* be taken on *i* and through *N* let the straight line *n* be drawn which is parallel to its correspondent *n′* let *m* and *n* meet in *S*, and *m′* and *n′* in *S′* If through *S* a straight line *l* be drawn parallel to *i*, its correspondent *l′* will pass through *S′* and will also be parallel to *i*, since the point at infinity on *i* corresponds to itself The corresponding pencils *S* and *S′* are therefore such that three rays *l*, *m*, *n* of the one are severally parallel to the three corresponding rays *l′*, *m′*, *n′* of the other, and consequently (see below, Art 104) the two pencils are equal Now let one of the planes be made to slide upon the other, without rotation, until *S′* comes into coincidence with *S*, then the two pencils will become concentric and since they are equal, every ray of the one will coincide with the ray corresponding to it in the other This being the case, every pair of corresponding points will be collinear with *S*, and the two figures will be homological, *S* being the centre of homology

98 Suppose that in a plane π is given a quadrangle *ABCD*, and in a second plane π' a quadrilateral *a′b′c′d′* By means of constructions analogous to those explained in Arts 94–96, the points and straight lines of the one plane can be put into unique correspondence

with those of the other, so that to any range in the first p
responds in the second plane a pencil projective with the sa
and to any pencil in the first plane corresponds in the secon
range projective with the said pencil. Two plane figures r
one another in this manner are called *correlative* or *reciprocal*

CHAPTER XI

PARTICULAR CASES AND EXERCISES

99 Two ranges are said to be *similar*, when to the points A, B, C, D, of the one correspond the points A', B', C', D', of the other, in such a way that the ratio of any two corresponding segments AB and $A'B'$, AC and $A'C'$, is a *constant* If this constant is *unity*, the ranges are said to be *equal*

Two similar ranges are projective, every anharmonic ratio such as $(ABCD)$ being equal to the corresponding ratio $(A'B'C'D')$ For suppose the bases of the two ranges to lie in the same plane (Fig 67) and let their point of intersection be denoted by P' when considered as a point belonging to u' and by Q when considered as a point belonging to u Let A, A' be any pair of corresponding points, P that point of u which corresponds to P', and Q' that point of u' which corresponds to Q Draw AA'' parallel to u', and $A'A''$ parallel to u

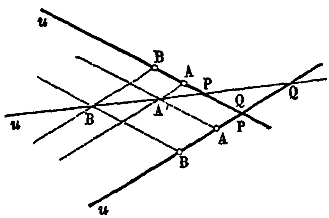


Fig 67

The triangles PQQ', PAA'' have the angles at Q and A equal and the sides about these equal angles proportionals, since by hypothesis

$$\frac{PQ}{P'Q'} = \frac{PA}{P'A'} = \frac{PA}{AA''}$$

Therefore the triangles are similar, and the angles QPQ' and APA'' are equal, and consequently the points P, Q', A'' are collinear If then the range ABC be projected upon PQ', by straight lines drawn parallel to u', we shall obtain the range $A''B''C''$ and from this last by projecting it upon

u' by straight lines drawn parallel to u, the range A' E may be derived

If $PQ=P'Q'$, i e if the straight line PQ' makes angles with the bases of the given ranges, the rang equal

To the point at infinity of u corresponds the point at i of u'

100 Conversely, *if the points at infinity I and I' projective ranges u and u' correspond to each other, the will be similar* For if (Fig 67) u be projected from I', from I (as in Art 85, left), two pencils of parallel rays formed, corresponding pairs of which intersect upon straight line u'' The segments $A''B''$ of u'' will be tional to the segments AB of u and also to the segment of u', consequently the segments AB of u will be prop to the segments $A'B'$ of u'

Otherwise if AA', BB', CC' are three pairs of sponding points, and I, I' the points at infinity, we h Art 73)

$$(ABCI) = (A'B'C'I'),$$

or (by Art 64), since I and I' are infinitely distant,

$$\frac{AC}{BC} = \frac{A'C'}{B'C'},$$

an equation which shows that corresponding segme proportional to one another

Examples If a flat pencil whose centre lies at a finite be cut by two parallel straight lines, two similar ranges of po be obtained

Any two sections of a flat pencil composed of parallel similar ranges

In these two examples the ranges are not only projective, in perspective in the first case the self corresponding poi infinity, in the second case it lies (in general) at a finite dist

101 Two flat pencils, whose centres lie at infinity, a jective and are called *similar*, when a section of the similar to a section of the other When this is the ca other two sections of the pencils will also be similar another

102 From the equality of the anharmonic ratios v clude that two equal ranges are projective (Art 79), a

conversely two projective ranges are equal (Art 73), when the corresponding segments which are bounded by the points of two corresponding triads ABC and $A'B'C'$ are equal, i e when $A'B'=AB$, $A'C'=AC$, (and consequently $B'C'=BC$)

Examples If a flat pencil consisting of parallel rays be cut by two transversals which are equally inclined to the direction of the rays, two directly equal ranges of points will be obtained *

If a flat pencil of non-parallel rays be cut by two transversals which are parallel to one another, and equidistant from the centre of the pencil, two oppositely equal ranges will be obtained *

103 Two similar ranges lying on the same base, and which have one self-corresponding point N at infinity, have also a second such point M, which is in general at a finite distance If AA', BB' are two pairs of corresponding points,

$$MA \quad MA'=AB \quad A'B'=\text{a constant}$$

To find M therefore it is only necessary to divide the segment AA' into two parts MA, MA' which bear to one another a given ratio

This ratio $MA \quad MA'$ is equal (Art 64) to the anharmonic ratio $(AA'MN)$ If its value is -1, the points $AA'MN$ are harmonic (Art 68), i e M is the middle point of AA', and similarly also that of every other corresponding segment BB', , in other words, the two ranges, which in this case are oppositely equal, are composed of pairs of points which lie on opposite sides of a fixed point M, and at equal distances from it

But if the constant ratio is equal to $+1$, i e if MA and MA' are equal in sign and magnitude, the point M will lie at infinity For since $(AA'MN)=1$, $(NMA'A)=1$ (Art 45), consequently the points M and N coincide

It follows also from the construction of Art 90 (Fig 66) that *two ranges on the same base, which have a single self-corresponding point lying at infinity, are directly equal*

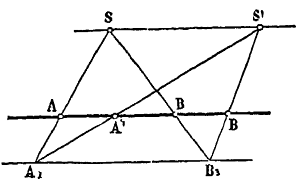


Fig 68

For if in Fig 66 the point M move off to infinity, the straight lines SS' and A_1B_1 become parallel to the given straight line u or u' on which the ranges lie (Fig 68), and as the triangles SA_1B_1 and $S'A_1B_1$ lie upon the same base and between the same parallels, the segments

* Imagine a moving point P to trace out a range ABC and its correspondent P' to trace out simultaneously the equal range $A'B'C'$ Then if P and P' move in the same direction, the two ranges are said to be *directly* equal if P and P' move in opposite directions the ranges are said to be *oppo tel e* l

which they intercept upon any parallel to the base are equal $AB=A'B'$, or *two corresponding segments are equal*, conseq $AA'=BB'$, *i e the segment bounded by a pair of corresponding is of constant length* We may therefore suppose the two ra have been generated by a segment given in sign and mag which moves along a given straight line, the one extremity the segment describes the one range, and the other extrem describes the other range

Conversely it is evident that if a segment AA', given in sig magnitude, slide along a given straight line, its extremities A will describe two directly equal (and consequently projective) which have a single self-corresponding point, lying at an distance

104 Two flat pencils are said to be *equal* when t elements of the one correspond the elements of the ot such a way that the angle included between any two of the first pencil is equal in sign and magnitude to the included between the two corresponding rays of the sec

It is evident that two such pencils can always be c two transversals in such a way that the resulting rang equal, but two equal ranges are always projective, the also *two equal flat pencils are always projective*

Conversely, *two projective flat pencils abcd and a'b'c'd' be equal if three rays abc of the one make with each other which are equal respectively to those which the three corresp rays make with each other*

This theorem may be proved by cutting the two p by two transversals in such a way that the sections and $A'B'C'$ of the groups of rays abc and $a'b'c'$ may be The projective ranges so formed will be equal (Art 102) sequently also the other corresponding angles ad and $a'd'$ the given pencils must be equal to one another

105 Since two equal forms (ranges or flat penci always projective with one another, it follows that if a or a flat pencil be placed in a different position in without altering the relative position of its elements, th in its new position will be projective with regard to the form in its original position

106 Consider two equal pencils $abcd$ and $a'b'c'd'$ same plane or in parallel planes, and suppose a ray one pencil to revolve about the centre and to descri

pencil, then the corresponding ray of the other pencil w
describe that other pencil, by revolving about its cent
This revolution may take place in the *same* direction
that of the first ray, or it may be in the *opposite* directio
in the first case the pencils are said to be *directly equal*, and
the second case to be *oppositely equal* to one another

In the first case the angles aa', bb', cc', are evidently
equal, in sign as well as in magnitude, consequently a p
of corresponding rays are either always parallel or ne
parallel

In the second case two corresponding angles are equal
magnitude, but of opposite signs If then one of the pen
be shifted parallel to itself until its centre coincides with t
of the other pencil, the two pencils, now concentric, will s
be projective (Art 105) and will evidently have a pair
corresponding rays united in each of the bisectors (inter
and external) of the angle included between two correspo
ing rays a and a' It follows that these rays are also
bisectors of the angle included between any other pair
corresponding rays If the first pencil be now replaced in
original position, so that the two pencils are no longer c
centric, we see that *there are in each pencil two rays, each
which is parallel to its correspondent in the other pencil, and t
two rays are at right angles to each other,* since they are para
to the bisectors of the angle between any pair of correspo
ing rays

107 *If two flat pencils abcd and a'b'c'd' are projective,
if the angles aa', bb', cc' included by three pairs of corresponding r
are equal in magnitude and of the same sign, then the angle
included by any other pair of corresponding rays will have the s
sign and magnitude*

For if we shift the first pencil parallel to itself unti
becomes concentric with the second, and then turn it about
common centre through the angle aa', the rays a, b, c will co
cide with the rays a', b', c' respectively The two pencils, wh
are still projective (Art 105), have then three self-correspo
ing rays, consequently (Art 82) every other ray will coinc
with its correspondent If now the first pencil be moved b
into its original position, the angle dd' will be equal to aa'

108 As the angles aa', bb', cc' of two directly eq

pencils are equal to one another, such pencils, when concen
and lying in the same plane, may be generated by the rota
of a constant angle aa' round its vertex O, supposed fixed,
one arm a traces out the one pencil, while the other arm
traces out the other pencil.

Conversely, if an angle of constant magnitude turn ro
its vertex, its arms will trace out two (directly) equal
therefore projective pencils. Evidently these pencils have
self-corresponding rays.

A transversal cutting these two pencils determines
itself two collinear ranges having no self-corresponding poi

What has been said in Arts 104–108 with respect to two pe
in a plane might be repeated without any alteration for the ca
two axial pencils in space

109 (1) Let ABC , $A'B'C'$ be two projective ranges
upon the same base, and let them, by means of the pencils ab
$a'b'c'$, be projected from different points U, U' Let i, j' be
rays passing through U, U' respectively, which are parallel to
given base, and let i', j be the rays corresponding to them
points I', J in which these last two rays cut the given base
be those points which correspond to the point at infinity (I
the base, according as that point is regarded as belonging
range ABC or to the range $A'B'C'$

The fact that the two corresponding groups of points are
jectively related gives an equation between the anharmonic r
from which we deduce (as in Art 74)

$$JA \cdot I'A' = JB \cdot I'B' = \text{a constant},$$

i e the product $JA \cdot I'A'$ is constant for every pair of points A, A

Let O be the middle point of the segment JI', and O' the
corresponding to O regarded as a point belonging to the first ran

Since the equation (1) holds for every pair of corresponding p
and therefore also for O and O', we have

$$JA \cdot I'A' = JO \cdot I'O',$$

or
$$(OA - OJ)(OA' - OI') + OJ(OO' - OI') = 0,$$

or since
$$OI' = -OJ,$$

$$OA \cdot OA' - OI'(OA - OA' + OO') = 0$$

Let us now enquire whether there are in this case any
corresponding points If such a point exist, let it be denoted b
then replacing both A and A' in (3) by F, we have

$$OE^2 = OI' \cdot OO'$$

We conclude that when $OI' \cdot OO'$ is positive, i e when O doe

lie between I' and O', there are two self-corresponding points E F, lying at equal distances on opposite sides of O, and dividing segment $I'O'$ harmonically (Art 69)

When O lies between I' and O', there are no such points

When O' coincides with O, there is only one such point, viz point O itself

(2) Imagine each of the given ranges to be generated by a p moving always in one direction* If the one range is describe the order ABC, the other range will be described in the order $A'B$ this order may be the same as the first, or may be opposite to it

If the order of ABC is opposite to that of $A'B'C'$, the same wil the case with regard to the order of IJA and that of $I'J'A'$, and a with regard to the finite segment JA and the infinite segment J *i e.* the finite segments JA and $I'A'$ have the same sign In sequence therefore of equation (2), JO and $I'O'$ have the same s so that O does not fall between I' and O' (Fig 69 *a*), there are th fore two self-corresponding points And these will lie outside finite segment JI', since OE is a mean proportional between OI' OO'

If the order of ABC is the same as that of $A'B'C'$, we arrive similar manner at the clusion that JA and I and again JO and $I'O'$, opposite signs In this then, self corresponding p exist if O does not lie

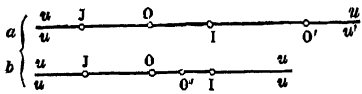


Fig 69

tween I' and O', that is, if O' lies between O and I' (Fig 69 *b*) these will lie within the segment JI', since OE is a mean proporti between OI' and OO'

(3) Suppose that there are two self-corresponding points E F (Fig 70), draw through E any straight line, on which take points S, S', and project o the ranges from S and other from S' The two pe which result are in perspec since they have a self-c sponding ray SES', accordi the corresponding rays SA $S'A'$, SB and $S'B'$, SF $S'F'$ will intersect in p lying on a straight lin

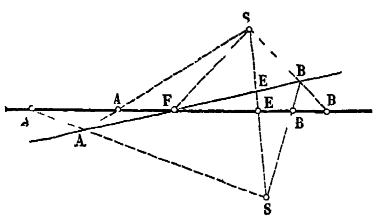


Fig 70

which passes through F

Let E'' be the point where this straight line u'' meets SS'

* [illegible]

$EFAA'$ and $EFBB'$ are the projections of $EE''SS'$ from the c A'' and B'' respectively, therefore $EFAA'$ and $EFBB'$ are proj with one another, thus the anharmonic ratio of the system com of any two corresponding points together with the two self-sponding points is constant

In other words *two projective forms which are superposed on the other, and which have two self-corresponding elements, are com of pairs of elements which give with two fixed ones a constant a monic ratio* *

(4) Next suppose that there are no self-corresponding poin that O lies between O' and I' (Fig 71) Draw from O a straig OU at right angles to the given base and make OU the geo mean between $I'O$ and OO', thus $I'UO'$ will be a right angle

Again, draw through U the straight line IUJ' parallel to the base, then the angle IUI' will be equal to JUJ', and the OUO' will be equal to $OI'U$ and therefore to IUI' Thus in the two projective pencils which project the two given ranges from U, the angles IUI', JUJ', OUO' included by three pairs of corresponding rays are all equal, consequently (Art 107) the angles AUA', BUB', are also all equal to them and to one anoth are all measured in the same direction †

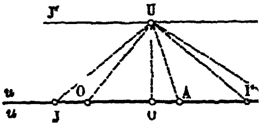


Fig 71

Thus *two collinear ranges which have no self-corresponding can always be regarded as generated by the intersection of th line with the arms of an angle of constant magnitude which always in the same direction, about its vertex*

110 We have seen (Art 84) the general solution of the p Given three pairs of corresponding elements of two projecti dimensional forms, to construct any desired number of pairs other words, to construct the element of the one form which sponds to a given element of the other The solution of the f particular cases is left as an exercise to the student

1 Suppose the two forms to be two ranges u and u' whic different bases, and let the given pairs of elements be

(a) P and P', Q and Q' ‡, I and I',

* The above construction gives the solution of the problem Given A, A and B B of corresponding points and one of the self correspondi E to find the other self corresponding point

† CHASLES *loc cit* p 119

‡ P, P, Q, Q', I, I, J, J' have the same meaning as in Art 84, A, any given points

(b)	P and P',	A and A',	B and B',
(c)	I and I',	J and J',	P and P',
(d)	I and I',	J and J',	A and A',
(e)	I and I',	P and P',	Q and Q',
(f)	I and I',	P and P',	A and A',
(g)	I and I',	A and A',	B and B'

2 Solve problems (d) and (g), supposing the ranges to be collinear

3 Solve the problems correlative to (a) and (b) when the two given forms are two non concentric pencils

4 Suppose one of the pencils to have its centre at infinity

5 Suppose both the pencils to have their centres at infinity

111 He may also prove for himself the following proposition

If the three vertices A, A', A'' of a variable triangle slide respectively on three fixed straight lines u, u', u'' which meet in a point, while two of its sides $A'A''$, $A''A$ turn respectively round two fixed points O and O', then will also the third side AA' always pass through a fixed point O'', collinear with O and O'

It is only necessary to show that the points A, A', A'' in moving describe three ranges which are two and two in perspective Or the theorem of Art 16 may be applied to two positions of the variable triangle

This proposition proved, the following corollary may be at once deduced

If the four vertices A, A', A'', A''' of a variable quadrangle slide respectively upon four fixed straight lines which all pass through the same point O, while three of its sides AA', $A'A''$, $A''A'''$ turn respectively round three fixed points C', B''', B', then will the fourth side $A'''A$ and the diagonals AA'', $A'A'''$ pass respectively through three other fixed points C''', C'', B'', which are determined by the three former ones The six fixed points are the vertices of a complete quadrilateral, i e they lie three by three on four straight lines (Fig 72)

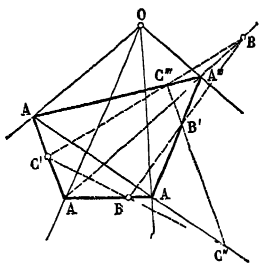


Fig 72

In a similar manner may be deduced the analogous corollary relating to a polygon of n vertices

112 THEOREM *If a triangle $O_1O_2O_3$ circumscribes another triangle $U_1U_2U_3$, there exist an infinite number of triangles each of which is circumscribed about the former and inscribed in the latter* (Fig 73)

The two pencils

$$O_2(U_1, U_2, U_3 \quad) \text{ and } O_3(U_1, U_2, U_3 \quad)$$

obtained by projecting the range U_2U_3 from O_2 and from evidently in perspective Similarly the pencils

$$O_1(U_1, U_2, U_3 \quad) \text{ and } O_3(U_1, U_2, U_3 \quad)$$

obtained by projecting the range U_1U_3 from O_1 and from in perspective Therefore the pencils

$$O_1(U_1, U_2, U_3 \quad) \text{ and } O_2(U_1, U_2, U_3 \quad)$$

are projective (Art 41), but the rays O_1U_3 and O_2U_3 co therefore (Art 62) the pencils are in perspective, and their sponding rays intersect in pairs on U_1U_2.

There are then three pencils O_1, O_2, O_3, which are two and two in perspective, corresponding rays of the first and second, second and third, third and first, intersecting in pairs on the straight lines U_1U_2, U_2U_3, U_3U_1 respectively This shows that every triad of corresponding rays will form a triangle which is circumscribed about the triangle $O_1O_2O_3$, and inscribed in the triangle $U_1U_2U_3$*

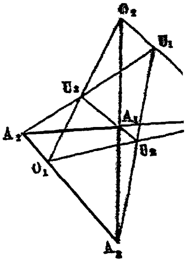


Fig 73

113 THEOREM *A variable straight line turning about a fixed point U cuts two fixed straight line u′ in A and A′ respectively, if S, S′ are two fixed points with uu′, and SA, S′A′ be joined, the locus of their point of int M will be a straight line* †

To prove this, we observe that the points A and A' tr two ranges in perspective with one another, and that conseque pencils generated by the moving rays SA, $S'A'$ are in per (Arts 41, 80)

The demonstration of the correlative theorem is propose exercise to the student

114 THEOREM *U, S, S′ are three collinear points, a tra turning about U cuts two fixed straight lines u and u′ in A respectively, if SA, S′A′ be joined, their point of intersection describe a straight line passing through the point uu′* ‡

The proof is analogous to that of the preceding theore

The proposition just stated may also be enunciated as follo *If the three sides of a variable triangle AA′M turn respective three fixed collinear points U, S, S′, while two of its vertice*

* STEINER loc cit, p 85 § 23, II Collected Works vol 1 p 297

† PAPPUS, loc cit, book VII props 123, 139, 141, 143, CHASLES, pp 241, 242

‡ CHASLES, loc cit, p 242

slide respectively upon two fixed straight lines u , u′, then will the third vertex M also describe a straight line *

In a like manner may be demonstrated the more general theorem

If a polygon of n sides displaces itself in such a manner that each of its sides passes through one of n fixed collinear points, while n − 1 of its vertices slide each on one of n − 1 fixed straight lines, then will also the remaining vertex, and the point of intersection of any two non consecutive sides, describe straight lines †

The correlative proposition is indicated in Art 85

115 PROBLEM *Given a parallelogram ABCD and a point P in its plane, to draw through P a parallel to a given straight line EF also lying in the plane, making use of the ruler only*

First Solution—Let *E* and *F* (Fig 74) be the points where the given straight line is cut by *AB* and *AD* respectively On *AC* take any point *K*, join *EK*, meeting *CD* in *G*, and *FK*, meeting *BC* in *H*

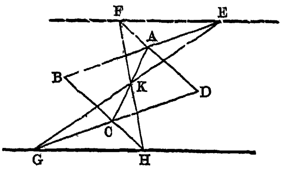


Fig 74

The triangles *AEF* , *CGH* are homological (Art 18), since *AC* , *EG* , *FH* meet in the same point *K*, and the axis of homology is the straight line at infinity, since the sides *AE* , *AF* of the first triangle are parallel respectively to the corresponding sides *CG*, *CH* of the second Therefore also the remaining sides *EF* and *GH* are parallel to one another ‡

The problem is thus reduced to one already solved (Art 86), viz given two parallel straight lines *EF* and *GH*, to draw through a given point *P* a parallel to them

Second Solution §—Produce (Fig 75) the sides *AB*, *BC*, *CD*, *DA* and a diagonal *AC* of the given parallelogram to meet the given straight line *EF* in *E*, *F*, *G*, *H*, *I* respectively, and join *EP*, *GP* Through *I* draw any straight line cutting *EP* in *A′* and *GP* in *C′*, and join *HA′*, *FC′* , if these meet in *Q*, then will *PQ* be the required straight line

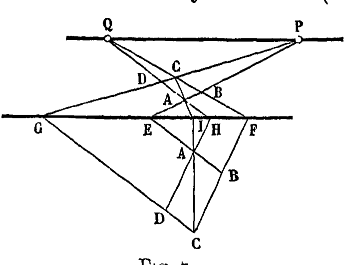


Fig 75

For if *B′* denote the point where *EP* cuts *FQ*, and *D′* the point

* This is one of Euclid s porisms See PAPPUS, *loc cit* preface to book VII

† This is one of the porisms of PAPPUS *loc cit* , preface to book VII

‡ PONCELET *Propriétés projectives* Art 198

§ LAMBERT, *Freie Perspective* (Zurich, 1774), vol ii p 169

where GP cuts HQ, the parallelograms $ABCD$ and $A'B'C$ homological, EF being the axis of homology The point sponds to the point of intersection of AB and CD, and the to that of BC and AD, therefore PQ corresponds to the lin finity in the first figure; accordingly it is the vanishing lin second figure, and consequently PQ is parallel to EF (Art. 18

116 PROBLEM *Given a circle and its centre, to draw a dicular to a given straight line, making use of the ruler only.*

Draw two diameters AC, BD of the circle (Fig 76), th $ABCD$ is then a rectangle Accordingly, if any point K be the circumference, then by means of the last proposition (Art 115) a parallel KL can be drawn to the given straight line EF If the point L where this parallel again meets the circumference be joined to the other extremity M of the diameter through K, then evidently LM will be perpendicular to KL, and therefore also to the given straight line

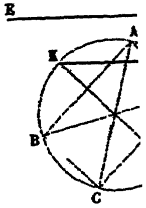


Fig 76

117 PROBLEM *Given a segment AC and its point of bisection B, to divide BC into n equal parts, ma of the ruler only*

Construct a quadrilateral $ULDN$ (Fig 77) of which one opposite sides DL, NU meet in A, the other pair LU, DN of which one diagonal DU passes through B, the other diag will be parallel to AC (Art 59), and will be bisected in M

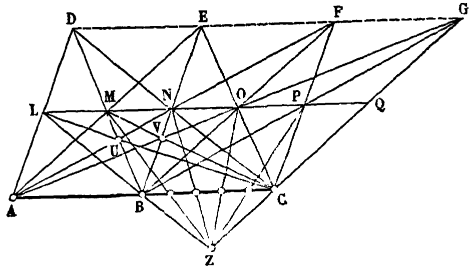


Fig 77

Now construct a second quadrilateral $VMEO$ which sati same conditions as the first, and which moreover has same conditions as the first, and which moreover has extremity and N for middle point of that diagonal which to AC To do this it is only necessary to join AM and BN, in E, and to join CE, this last will cut LN produced in a

such that $NO=MN=LM$ Now construct a third quadrilateral analogous to the first two, and which has N for an extremity and O for middle point of that diagonal which is parallel to AC If P is the other extremity of this diagonal, then $OP=NO=MN=LM$ Proceed in a similar manner, until the number of the equal segments LM, MN, NO, OP, is equal to n

If PQ is the segment last obtained, join LB, meeting QC in Z, the straight lines which join Z to the points M, N, O, P, will divide BC into n equal parts *

118 The following problems, to be solved by aid of the ruler only, are left as exercises to the student

Given two parallel straight lines AB and u, to bisect the segment AB (Art 59)

Given a segment AB and its point of bisection C, to draw through a given point a parallel to AB (Art 59)

Given a circle and its centre, to bisect a given angle (Art 60)

Given two adjacent equal angles AOC, COB, to draw a straight line through O at right angles to OC (Art 60)

119 THEOREM *If two triangles ABC, $A'B'C'$, lying in different planes σ, σ', are in perspective, and if the plane of one of them be made to turn round $\sigma\sigma'$, then the point O in which the rays AA', BB', CC' meet will change its position, and will describe a circle lying in a plane perpendicular to the line $\sigma\sigma'$* †

Let D, E, F (Fig 78) be the points of the straight line $\sigma\sigma'$ in which the pairs of corresponding sides BC and $B'C'$, CA and $C'A'$, AB and $A'B'$ meet respectively (Art 18) First consider the planes of the triangles to have any given definite position, and let O be the centre of projection for that position Through O draw OG, OH, OK parallel respectively to the sides of the triangle $A'B'C'$, as these parallels lie in the same plane (parallel to σ') they will meet the plane σ in three points G, H, K of the line $\pi\sigma$

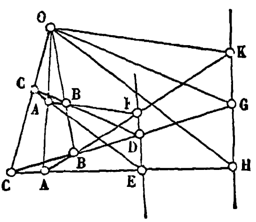


Fig 78

Now suppose the plane σ' together with the triangle $A'B'C'$ to turn round the line $\sigma\sigma'$ The range $BCDG$ is in perspective with the range $B'C'DG'$ (where G' denotes the point at infinity on $B'C'$), therefore the anharmonic ratio $(BCDG)$ is equal to the anharmonic ratio $(B'C'DG')$, i e to the simple ratio $B'D$ $C'D$ (Art 64), which is

* These and other problems, to be solved by aid of the ruler only, will be found in the work of LAMBERT quoted above

† CHASLES, *loc cit*, Arts 368, 369 This proposition has already been proved by a different method in Art 22

constant Since then B, C, D are fixed points, G must a fixed and invariable point (Art. 65). From the similar t OBG, $B'BD$

$$OG \quad B'D \quad BG \quad BD,$$

$$OG = \frac{B'D \quad BG}{BD},$$

i e OG is constant. The point O therefore moves on a spher centre is G and whose radius is the constant value just found

In a similar manner it may be shown that O moves upon two other spheres having their centres at H and K respective

Since then the point O must lie simultaneously on several its locus must be a circle, whose plane is perpendicular to of centres of the spheres, and whose centre lies upon this sam

This line GHK is the line of intersection of the planes and is consequently parallel to $\sigma\sigma'$ (since π and σ' are parallel it is the vanishing line of the figure σ, regarded as the pe image of the figure σ' (Art. 13)

120 THEOREM *Two concentric projective pencils lying in plane, which have no self-corresponding rays, may be regard perspective image of two directly equal pencils* *

Let O be the common centre of the two pencils Cut t transversal s, thus forming two collinear projective ranges and $A'B'C'$ which have no self-corresponding points Draw s any plane σ', we can determine in this plane (Art 109) such that the segments AA', BB', CC', subtend at it a angle, thus if the two ranges be projected from U as ce directly equal pencils will be obtained Now let the eye be any point of the straight line OU, and let the given penci jected from this point as centre on to the plane σ' In thi new pencils will be formed, and these are precisely the tw equal pencils mentioned in the enunciation

* CHASLES, *loc cit* Art 180

CHAPTER XII

INVOLUTION

121. CONSIDER two projective flat pencils (Fig 79) having
common centre O, let them be cut in corresponding points
the transversals u and u', thus giving two projective rang
ABC and $A'B'C'$, and let u''
the straight line on which the pa
of lines AB' and $A'B$, (Art 85, le
intersect Through O draw any r
(not a self-corresponding ray), it w
cut u and u' in two non-correspondi
points A and B' and will meet u''
a point of the line $A'B$ To the ray

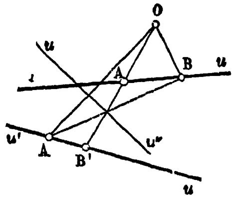


Fig 79

of the first pencil corresponds accordingly the ray OA' of t
second, and to the ray OB' of the second pencil correspon
the ray OB of the first In other words, to the ray OA or O
correspond two different rays OA', OB according as the fi
ray is regarded as belonging to the first pencil or to
second For the line $A'B$ must cut AB' on u'', and cann
pass through O so long as this point does not lie on u''
see then that

In two superposed projective forms (of one dimension) th correspond, in general, to any given element two different eleme according as the given element is regarded as one belonging to first or to the second form*

We say *in general*, because in what precedes it has b
assumed that O does not lie upon u''

* We say two forms, because the reasoning which we have made use of in
case of two concentric flat pencils may equally well be applied in the case of
collinear ranges, and of two axial pencils having a common axis The same re
may be arrived at by cutting the two flat pencils by a transversal, and by
jecting them from [illegible]

122 But in the case where O lies upon s'' (Fig
ray be drawn through O to cut u and u' in A and B
tively, then will also $A'B$ pass through O, in other w
the ray OA or OB' corresponds
the same ray OA' or OB
This property may be expressed by saying that *the two rays correspond doubly to one another*, or we may say that *the two rays are conjugate to one another*

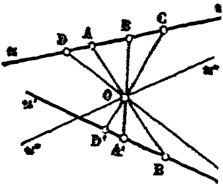


Fig 80.

Now suppose, reciprocally, that two concentric projective flat pencils have a pair of rays which correspond doubly to one another Cut th by two transversals u and u', and let A and B' de points where these transversals intersect one of t rays, then A' and B will denote the points wh intersect the other given ray The straight line locus of the points of intersection of the pairs of l as MN', $M'N$, formed by joining crosswise any t corresponding points of the ranges u, u' (Art 85), through O, since the lines AB', $A'B$ meet in that] now there be drawn through O any other ray, cu transversals say in C and D', then will $C'D$ also pas O, *i e* the rays OCD' and ODC' also correspond each other We conclude that

When two superposed projective forms of one dimensio that any one element has the same correspondent, to form it be regarded as belonging, then every element po property

123 This particular case of two superposed projec of one dimension is called *Involution* * We spe involution of points, of rays, or of planes, accordi elements are points of a range, rays of a flat pencil, of an axial pencil

In an involution, then, *the elements are conjuga another in pairs*, *i e* each element has its conju whichever of the two forms a given element be con

* DESARGUES, *Brouillon projet d'une atteinte aux evenements des re cone avec un plan* (Paris, 1639) edition POUDRA (Paris, 1864, vol

belong, the element which corresponds to it is the same, viz its conjugate. It follows from this that it is not necessary to regard the two forms as distinct, but that *an involution may be considered as a set of elements which are conjugate to one another in pairs*.

When AA', BB', CC', ... are said to form an involution, it is to be understood that A and A', B and B', C and C', ... are pairs of conjugate elements; moreover, any element and its conjugate may be interchanged, so that $AA'BB'CC'$ ... and $A'AB'BC'C$ ... are projective forms.

124. Since an involution is only a particular case of two superposed projective forms, *every section and every projection of an involution gives another involution*.*

Two conjugate elements of the given involution give rise to two conjugate elements of the new involution. It follows (Art. 18) that the figure homological with an involution is also an involution.

125. When two collinear projective ranges form an involution, there corresponds to each point (and consequently also to the point at infinity I or J') a single point (I' or J), *i.e. the two vanishing points coincide in a single point*. Let this point, the conjugate of the point at infinity, be denoted by O. The equation (1) of Art. 109 then becomes

$$OA \cdot OA' = \text{constant}.$$

In other words, an involution of points consists of pairs of points A, A' which possess the property that the rectangle contained by their distances from a fixed point O, lying on the base, is constant.† This point O is called the *centre* of the involution.

The self-corresponding elements of two forms in involution are called the *double elements* of the involution. In the case of the involution of points AA', BB', ... we have

$$OA \cdot OA' = OB \cdot OB' = \ldots = \text{constant}.$$

If this constant is positive, *i.e.* if O does not lie between two conjugate points, there are two double points E and F, such that

$$OE^2 = OF^2 = OA \cdot OA' = OB \cdot OB' = \ldots,$$

* DESARGUES, *loc. cit.*, p. 147.

† Ibid.

O therefore lies midway between E and F, and the se
EF divides harmonically each of the segments AA', B
(Art 69 [3]) Accordingly

If an involution has two double elements, these separat monically any pair of conjugate elements, or *An involution i up of pairs of elements which are harmonically conjugate with to two fixed elements*

If, on the other hand, the constant is negative, *i e* if between two conjugate points, there are no double point this case there are two conjugate points situated at distances from O and on opposite sides of it, such $OE = -OE'$, and

$$OE^2 = OE'^2 = -OE \cdot OE' = -OA \cdot OA'$$

If the constant is zero, there is only one double po but in this case there is no involution properly so For since the rectangle $OA \cdot OA'$ vanishes, one out of pair of conjugate points must coincide with O

126 The proposition that if an involution has two elements, these separate harmonically any pair of co elements, may also be proved thus

Let E and F be the double elements, A and A' any conjugate elements, since the systems $EFAA'$, $EFA'A$ a jective, therefore (Art 83) each of them is harmonic

The following is a third proof

Consider EAA' and $EA'A$ as two projective rang project them respectively from two points S and S' c with E (Fig 81) The projecting pencils $S(EAA'$ $S'(EA'A$) are in perspective (since they have a self-corresponding ray in $SS'E$), therefore the straight line which joins the point of intersection of SA and $S'A'$ to that of SA' and $S'A$ will contain the points of intersection of all pairs of corresponding rays, and will consequently meet the common base of the two ranges at the second point F But from the figure we see that we have complete quadrilateral, one diagonal of which, AA', is the other two in E and F, consequently (Art 56) EFA harmonic range

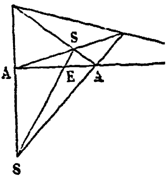


Fig 81

The proposition itself is a particular case of that proved in Art 109 (3) From this we conclude that the pairs of elements (points of a range, rays or planes of a pencil) which, with two fixed elements, give a constant anharmonic ratio, form two superposed projective forms, which become an involution in the case where the anharmonic ratio has the value -1 (Art 68)

127 *An involution is determined by two pairs of conjugate elements*

For let A, A' and B, B' be the given pairs If any element C be taken, its conjugate is determinate, and can be found as in Art 84, by constructing so that the form $A'A\ B'C'$ shall be projective with $AA'BC$ We then say that *the six elements* AA', BB', CC' *are in involution*, *i e* they are three pairs of an involution

Suppose that the involution with which we have to deal is an involution of points Take any point G (Fig 82) outside the base, and describe circles round GAA' and GBB', if H is the second point in which these circles meet, join GH, and let it cut the base in O Since $GHAA'$ lie on a circle,

$$OG\ \ OH = OA\ \ OA',$$

and since $GHBB'$ lie on a circle,

$$OG\ \ OH = OB\ \ OB',$$

$$OA\ \ OA' = OB\ \ OB'$$

O is therefore the centre of the involution determined by the

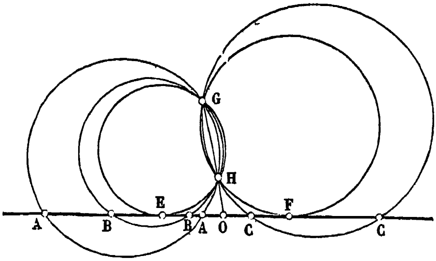


Fig 82

pairs of points A, A' and B, B' If any other circle be drawn through G and H, and cut the base in C and C', we have

$$OG\ \ OH = OC\ \ OC',$$

$$OC\ \ OC' = OA\ \ OA' = OB\ \ OB',$$

and C, C' are therefore a pair of conjugate points of the involution In other words, the circle which passes through two

conjugate points C, C' or D, D' and through one of the p
G, H always passes through the other Accordingly
The pairs of conjugate points of the involution are the poi intersection of the base with a series of circles passing throug points G and H

128 From what precedes it is evident that if the involu has double points, these will be the points of contact o base with the two circles which can be drawn to pass thr G and H and to touch the base It has already been (Art 125) that these points are harmonically conjugate regard to A and A', and also with regard to B and B' Consequently (Art 70) *the involution has double points when one of the pairs AA', BB' lies entirely within or entirely without the other, i e when the segments AA' and BB' do not overlap* (Fig 82), *and the involution has no double points when one pair is alternate to the other, i e when the segme and BB' overlap* (Fig 83)*

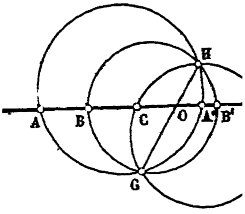


Fig 83.

In the first case, the involution (as already seen) consi an infinite number of pairs of points which are harmon conjugate with regard to a pair of fixed points

In the second case, on the other hand, the involuti traced out on the base by the arms of a right angle which revolves about its vertex For since (Fig 84) the segments AA' and BB' overlap, the circles described on AA' and BB' respectively as diameters will intersect in two points G and H which lie symmetrically with regard to the GH being perpendicular to the base, which bisects it the centre of the involution It follows that

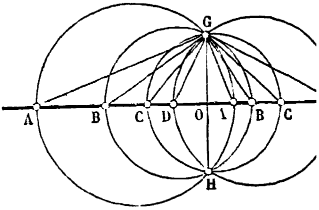


Fig 84.

* An involution of the kind which has double points is often called a *hyp* involution, one of the kind which has no double points being called an involution

$$OG^2 = OH^2 = AO \cdot OA' = BO \cdot OB',$$

and that all other circles passing through G and H cutting the base in the other pairs CC', DD', of the invo tion will have their centres also on the base, and will h CC', DD', as diameters If then we project any of segments AA', BB', CC', from G (or H) as centre, we sl obtain in each case a right angle AGA', BGB', CGC' (or AHA', BHB', CHC',)

We conclude that when an involution of points AA', BB has no double points, *i e* when the rectangle $OA \cdot OA'$ is ec to a negative constant $-k^2$, each of the segments AA', BB subtends a right angle at every point on the circumferenc a circle of radius k, whose centre is at O and whose plan perpendicular to the base of the involution

This last proposition is a particular case of that of Art 109 If then an angle of constant magnitude revolve in its plane abou vertex, its arms will determine on a fixed transversal two projec ranges, which are in involution in the case where the angle is a r angle

id

129 Consider an involution of parallel rays, these meet in a p at infinity, and the straight line at infinity is a ray of the involut The ray conjugate to it contains the centre of the involution of pc which would be obtained by cutting the pencil by any transversal may therefore be called the *central ray* of the given involution reciprocally, we project an involution of points by means of par rays, these rays will form a new involution, whose central ray pa through the centre of the given involution

When one involution is derived from another involution by m of projections or sections (Art 124), the double elements of the always give rise to the double elements of the second

Take

130 Since in an involution any group of elements is projective the group of conjugate elements, it follows that if any four point the involution be taken, their anharmonic ratio will be equal to of their four conjugates In the involution AA', BB', CC', groups of points $ABA'C'$ and $A'B'AC$, for example, will be project therefore

$$\frac{AA'}{BA'} \cdot \frac{AC'}{BC'} = \frac{A'A}{B'A} \cdot \frac{A'C}{B'C},$$

whence

$$AB' \cdot BC' \cdot CA' + A'B \cdot B'C \cdot C'A = 0$$

Conversely, if this relation hold among the segments determine six collinear points $AA'BB'CC'$, these will be three conjugate pai

an involution For the given relation shows that the anhar ratios $(ABA'C')$ and $(A'B'AC)$ are equal to one another, the g $ABA'C'$ and $A'B'AC$ are therefore projective. But A and A' spond doubly to each other, therefore (Art. 122) AA', BB', C three conjugate pairs of an involution.

131 THEOREM *The three pairs of opposite sides of a complete quadrangle are cut by any transversal in three pairs of conjugate points of an involution* *

Let $QRST$ (Fig 85) be a complete quadrangle, of which the pairs of opposite sides RT and QS, ST and QR, QT and RS are cut by any transversal in A and A', B and B', C and C' respec-

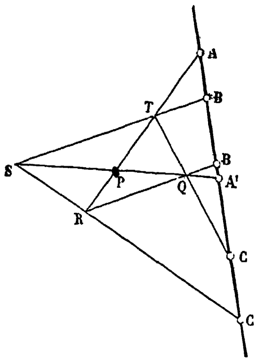


Fig 85

tively If P is the point of intersection of QS and RT, then $ATPR$ is a projection of $ACA'B'$ from Q as centre, and $ATPR$ is also a projection of $ABA'C'$ from S as centre, therefore the group $ACA'B'$ is projective with $ABA'C'$, and therefore (Art 45) with $A'C'AB$ And since A and A' correspond doubly to one another in the projective groups $ACA'B'$

CORRELATIVE THEOREM. *straight lines which conne point with the three pairs of site vertices of a complete q lateral are three pairs of con rays of an involution.*

Let $qrst$ (Fig 86) be plete quadrilateral, of whi pairs of opposite vertices rt st and qr, qt and rs are pr from any centre by the ray a', b and b', c and c' respec

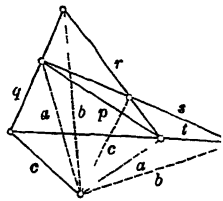


Fig 86

Let p be the straight lin joins the points qs and pencils $atpr$ and $aca'b'$ ar spective (their correspond intersect in pairs on q), s $atpr$ and $aba'c'$ are in tive (their correspondin intersect in pairs on s) pencil $atpr$ is therefore o projective with each pencils $aca'b'$ and aba

* DESARGUES, *loc cit*, p 171

and $A'C'AB$, it follows (Art 122) that AA', BB', CC' are three conjugate pairs of an involution

therefore $aca'b'$ is projective with $aba'c'$ or (Art 45) with $a'c'ab$ And since a and a' correspond doubly to one another in the pencils $aca'b'$ and $a'c'ab$, it follows (Art 122) that aa', bb', cc' are three pairs of conjugate rays of an involution

The theorem just proved may also be stated in the following form

If a complete quadrangle move in such a way that five of its sides pass each through one of five fixed collinear points, then its sixth side will also pass through a fixed point collinear with the other five, and forming an involution with them

The theorem just proved may also be stated in the following form

If a complete quadrilateral move in such a way that five of its vertices slide each on one of five fixed concurrent straight lines, then its sixth vertex will also move on a fixed straight line, concurrent with the other five, and forming an involution with them

132 By combining the preceding theorem (left) with that of Art 130, we see that

If a transversal be cut by the three pairs of opposite sides of a complete quadrangle in A and A', B and B', C and C' respectively, these determine upon it segments which are connected by the relation

$$AB' \cdot BC' \cdot CA' + A'B \cdot B'C \cdot C'A = 0 \text{*}$$

133 In the theorem of Art 131 (right) let U and U', V and V', W and W' denote respectively the opposite vertices rt and qs, st and qr, qt and rs of the quadrilateral $qrst$, and let AA', BB', CC' denote respectively the points of intersection of the rays aa', bb', cc' with an arbitrary transversal With the help of Art 124 the following proposition may be enunciated

If the three pairs UU', VV', WW' of opposite vertices of a complete quadrilateral be projected from any centre upon any straight line, the six points AA', BB', CC' so obtained will form an involution

Suppose now, as a particular case of this, that the centre of projection G is taken at one of the two points of intersection of the circles described on UU', VV' respectively as diameters Then AGA' and BGB' are right angles, and therefore also (Art 128) CGC' is a right angle, therefore the circle on WW' as diameter will also pass through G Hence the three circles which have for diameters the three diagonals of a complete quadrilateral pass all through the same two

* Pappus, *loc cit*, book VII prop 130

points, that is, they have the same radical axis. The cent these circles lie in a straight line, hence

The middle points of the three diagonals of a complete quadr are collinear *

134 The proposition of Art 131 (left) leads immediately to the

Construction for the sixth point C' of an involution of which five points A, A', B, B', C are given.

For draw through C (Fig 85) an arbitrary straight line, on which take any two points Q and T, and join AT, BT, $A'Q$, $B'Q$, if AT, $B'Q$ meet in R, and BT, $A'Q$ in S, the straight line RS will cut the base of the involution in the required point C'

The proposition of Art (right) leads immediately to

Construction for the six c' of an involution of whi rays a, a', b, b', c are given.

For take on c (Fig 86) a trary point, through which any two straight lines q and join the point ta to q the point tb to qa', if the lines be called r, s respe then the straight line con the centre of the pencil w point rs is the required ray

If, in the preceding problem (left), the point C lies at infi conjugate is the centre O of the involution. In order then *the centre of an involution of which two pairs AA', BB' of conjugate points are given*, we construct (Fig 87) a complete quadrangle $QSTR$ of which one pair of opposite sides pass respectively through A and A', another such pair through B and B', and which has a fifth side parallel to the base, the sixth side will then pass through the centre O

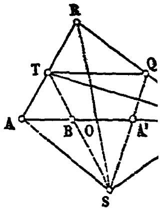


Fig 87

The sixth point C' which, together with five given points $AA'BB'C$, forms an involution, is completel mined by the construction, there is only one point C' which p the property on which the construction depends (Art 127) may be otherwise seen by regarding C' as given by the e $(AA'BC)=(A'AB'C')$ between anharmonic ratios, for it is (Art 65) that there is only one point C' which satisfies this eq

135 The theorem converse to that of Art 131 is t lowing

If a transversal cut the sides of a triangle RSQ (Fig three points A', B', C' which, when taken together with thr points A, B, C lying on the same transversal, form three co

* CHASLES, *loc cit*, Arts 344 345, GAUSS, Collected Works, vol iv

pairs of an involution, then the three straight lines RA, SB, QC meet in the same point

To prove it, let RA, SB meet in T, and let TQ meet the transversal in C_1 Applying the theorem of Art 131 (left) to the quadrangle $QRST$, we have

$$(AA'BC_1) = (A'AB'C')$$

But by hypothesis

$$(AA'BC) = (A'AB'C'),$$
$$(AA'BC_1) = (AA'BC),$$

consequently (Art 54) C_1 coincides with C, *i e* QC passes through T

The correlative theorem is

If a point S be joined to the vertices of a triangle rsq (Fig 86) *by three rays a', b', c' which, when taken together with three other rays a, b, c passing also through S, form three conjugate pairs of an involution, then the points ra, qb, sc lie on the same straight line t*

take

136 Take again the figure of the complete quadrangle $QRST$ whose three pairs of opposite sides are cut by a transversal in A and A', B and B', C and C' Let (Fig 88) SQ and RT meet in R', QR and ST in S', RS and QT in Q'

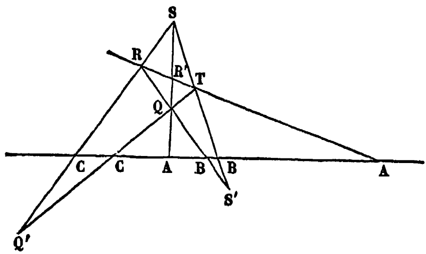


Fig 88

Consider the triangle RSQ, on each of its sides we have a group of four points, viz

$$SQR'A', \quad QRS'B', \quad RSQ'C'$$

The projections of these from T on the transversal are

$$BCAA', \quad CABB', \quad ABCC'$$

The product of the anharmonic ratios of these last three groups is

$$\left(\frac{BA}{CA} \quad \frac{BA'}{CA'}\right)\left(\frac{CB}{AB} \quad \frac{CB'}{AB'}\right)\left(\frac{AC}{BC} \quad \frac{AC'}{BC'}\right),$$

or
$$-\frac{CA' \cdot AB' \cdot BC'}{BA' \cdot CB' \cdot AC'},$$
which (Art. 130) is equal to -1 Therefore

If any transversal meet the sides of a triangle, and if moreov any point as centre each vertex be projected upon the side oppos the groups of four points thus obtained on each of the sides of the t will be such that the product of their anharmonic ratios is equal

Conversely, *if three pairs of points $R'A'$, $S'B'$, $Q'C'$ be one on each of the sides of a triangle RSQ, such that the pro the anharmonic ratios $(SQR'A')$, $(QRS'B')$, $(RSQ'C')$ is equal then, if the straight lines RR', SS', QQ' are concurrent, th A', B', C' will be collinear, and conversely, if the points A', are collinear, the straight lines RR', SS', QQ' will be concu*

137 Suppose now the transversal to lie altoget infinity, then the anharmonic ratios $(SQR'A')$, $(QRS'I$ $(RSQ'C')$ become (Art. 64) respectively equal to SR' $QS' \cdot RS'$, and $RQ' \quad SQ'$, so that the preceding prop reduces to the following*

If the straight lines connecting the three vertices of a RSQ with any given point T meet the respectively opposi in R', S', Q', the segments which they determine on t connected by the relation
$$\frac{SR' \cdot QS' \cdot RQ'}{QR' \cdot RS' \cdot SQ'} = -1,$$
and conversely

If on the sides SQ, QR, RS respectively of a triang points R', S', Q' be taken such that the above relation ho will the straight lines RR', SS', QQ' meet in one point T

138 Repeating this last theorem for two points T' we obtain the following

If the two sets of three straight lines which connect the v a triangle RSQ with any two given points T' and T'' respectively opposite sides in R', S', Q' and R'', S'', Q'', the product of the anharmonic ratios $(SQR'R'')$, $(QRS'$ $(RSQ'Q'')$ be equal to $+1$

[For each of the expressions
$$\frac{SR' \cdot QS' \cdot RQ'}{QR' \cdot RS' \cdot SQ'}, \quad \frac{SR'' \cdot QS'' \cdot RQ''}{QR'' \cdot RS'' \cdot SQ''}$$

* CEVA's theorem See his book, *De lineis rectis se invicem secanti constructio* (Mediolani 1678), I 2 Cf MÖBIUS, *Baryc Calc* § 198

is equal to -1, and the required result follows on dividing one of them by the other]

139 Considering again the triangle QRS (Fig 88), and taking the transversal to be entirely arbitrary, let ST, QT be taken so as to be parallel to QR, RS respectively Then the figure $QRST$ becomes a parallelogram, the points S' and Q' pass to infinity, and R' (being the point of intersection of the diagonals QS, RT) becomes the middle point of SQ Consequently (Art 64) the anharmonic ratios $(SQR'A')$, $(QRS'B')$, $(RSQ'C')$ become equal respectively to $-(QA' \; SA')$, $(RB' \; QB')$, and $(SC' \; RC')$ Thus *

If a transversal cut the sides of a triangle RSQ in A', B', C' respectively, it determines upon them segments which are connected by the relation

$$\frac{QA' \; RB' \; SC'}{SA' \; QB' \; RC'} = 1,$$

and conversely

If on the sides SQ, QR, RS respectively of a triangle points A', B', C' be taken such that the above relation holds, then will these three points be collinear

140 Repeating the last theorem of the preceding Article for two transversals, we obtain the following

If the sides of a triangle RSQ are cut by two transversals in A', B', C' and in A'', B'', C'' respectively, the product of the anharmonic ratios $(SQA'A'')$, $(QRB'B'')$, and $(RSC'C'')$ will be equal to $+1$

[For each of the expressions

$$\frac{QA' \; RB' \; SC'}{SA' \; QB' \; RC'}, \quad \frac{QA'' \; RB'' \; SC''}{SA'' \; QB'' \; RC''}$$

is equal to 1, dividing one by the other, the required result follows]

Reciprocally, *if on the sides of a triangle RSQ three pairs of points $A'A''$, $B'B''$, $C'C''$ be taken such that the product of the anharmonic ratios $(SQA'A'')$, $(QRB'B'')$, $(RSC'C'')$ may be equal to $+1$, then, if the points A', B', C' are collinear, the points A'', B'', C'' will also be collinear, and if the lines RA', SB', QC' are concurrent, the lines RA'', SB'', QC'' will also be concurrent*

141 It has been shown (Art 122) that if two projective ranges

* Theorem of MENELAUS, *Sphaerica*, iii 1 Cf MÖBIUS *loc cit*

(ABC) and ($A'B'C'$), lying in the same plane, are projected the point of intersection of a pair of lines such as AB' and $A'B$, and $A'C$, or BC' and $B'C$, the projecting rays form an involu The theorems correlative to this are as follows

Given two projective, but not concentric, flat pencils (abc) ($a'b'c'$) lying in the same plane, if they be cut by the straight which joins a pair of points such as ab' and $a'b$, ac' and $a'c$, c and $b'c$, the points so obtained form an involution.

Given two projective axial pencils ($\alpha\beta\gamma$) and ($\alpha'\beta'\gamma'$) w axes meet one another, if they be cut by the plane which is d mined by passing through a pair of lines such as $\alpha\beta'$ and $\alpha'\beta$, $\alpha\gamma'$ $\alpha'\gamma$, or $\beta\gamma'$ and $\beta'\gamma$, the rays so obtained form an involution.

Given two projective flat pencils (abc) and ($a'b'c'$) which concentric, but lie in different planes, if they be projected from point of intersection of a pair of planes such as ab' and $a'b$, ac $a'c$, or bc' and $b'c$, the projecting planes form an involution.

142 Particular Cases. All points of a straight line which pairs at equal distances on opposite sides of a fixed point on the form an involution, since every pair is divided harmonically b fixed point and the point at infinity

Conversely, if the point at infinity is one of the double points involution of points, then the other double point bisects the dis between any point and its conjugate If in such an i segments AA', BB' formed by any two pairs of conjugate p a common middle point, then will this point bisect also the se CC' formed by any other pair of conjugates

All rectilineal angles which have a common vertex, lie in the plane, and have the same fixed straight line as a bisector, form volution, since the arms of every angle are harmonically con with regard to the common bisector and the ray perpendicula through the common vertex

Conversely, if the double rays of a pencil in involution incl right angle, then any ray and its conjugate make equal angle either of the double rays If in such an involution the angles inc by two pairs of conjugate rays aa' and bb' have common bise these will be the bisectors also of the angle included by any othe of conjugate rays cc'

All dihedral angles which have a common edge and which ha same fixed plane as a bisector, form an involution, for the fa every angle are harmonically conjugate with regard to the fixed and the plane drawn perpendicular to it through the common ec

Conversely, if the double planes of an axial pencil in involutio at right angles to one another, then any plane and its conjugate equal angles with either of the double planes

CHAPTER XIII

PROJECTIVE FORMS IN RELATION TO THE CIRCLE

143 CONSIDER (Fig 89) two directly equal pencils $abcd$ and $a'b'c'd'$ in a plane, having their centres at O and O' respectively The angle contained by a pair of corresponding rays $aa', bb', cc',$ is constant (Art 106), the locus of the intersection of pairs of corresponding rays is therefore (Euc III 21) a circle passing through O and O' The tangent to this circle at O makes with OO' an angle equal to any of the angles OAO', OBO', OCO', &c , but this is just the angle which $O'O$ considered as a ray of the second pencil should make with the ray corresponding to it in the first pencil, therefore to $O'O$ or q' considered as a ray of the second pencil corresponds in the first pencil the tangent q to the circle at O

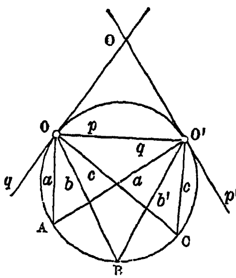


Fig 89

Imagine the circumference of the circle to be described by a moving point A, the rays AO, AO' or a, a' will trace out the two pencils As A approaches O, the ray AO' will approach OO' or q' and the ray AO will approach q, and in the limit when A is indefinitely near to O, the ray AO will coincide with q or the tangent at O This agrees with the definition of the tangent at O, as the straight line which joins two indefinitely near points of the circumference

Similarly, to the ray OO' or p considered as belonging to the first pencil corresponds the ray p' of the second pencil, the tangent to the circle at O'

144 Conversely, if any number of points A, B, C, D, on a circle be joined to two points O and O' lying on the same

circle, the pencils $O(A, B, C, D, \quad)$ and $O'(A, B, C, D, ..$ formed will be directly equal, since the angle AOB is equ $AO'B$, AOC to $AO'C$, BOC to $BO'C$, &c. But two pencils are always projective with one another (Art. 104 then the points $A, B, C,$ remain fixed, while the cent the pencil moves and assumes different positions on th cumference of the circle, the pencils so formed are all equ one another, and consequently all projective with one an The tangent at O is by definition the straight line which O to the point indefinitely near to it on the circle. It fo that in the projective pencils $O(A, B, C, \quad)$ and $O'(A, B,$ the ray of the first which corresponds to the ray $O'O$ second is the tangent at O

145 It has been seen (Art. 73) that in two projective four harmonic elements of the one correspond to four har elements of the other If then the four rays $O(A, B,$ form a harmonic pencil, the same is the case with reg the four rays $O'(A, B, C, D)$, whatever be the position point O' on the circle By taking O' indefinitely near we see that the pencil composed of the tangent at the chords AB, AC, AD will also be harmonic, so aga pencil composed of the chord BA, the tangent at B, a chords BC, BD will be harmonic, &c.

When this is the case, *the four points A, B, C, D of th are said to be harmonic* *

146 *The tangents to a circle determine upon any pair of fixed tangents two ranges which are projective with one another*

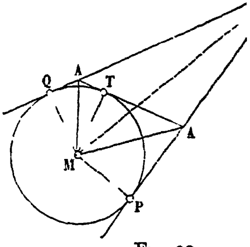


Fig 90

Let M (Fig 90) be the centre of the circle, PQ and $P'Q'$ a pair of fixed tangents, and AA' a variable tangent The part AA' of the variable tangent intercepted between the fixed tangents subtends a constant angle at M, for if Q, P', T are the po contact of the tangents respectively,

* STEINER *loc cit*, p 157, § 43 Collected Works, vol I. p 34

angle $AMA' = AMT + TMA'$

$$= \frac{1}{2} QMT + \frac{1}{2} TMP'$$

$$= \frac{1}{2} QMP' *$$

Accordingly, as the tangent AA' moves, the rays MA, MA' will generate two projective pencils (Art 108), and the points A, A' will trace out two projective ranges

Since the angle AMA' is equal to the half of QMP', it is equal to either of the angles QMQ', PMP' (denoting by P and Q' the same point, according as it is regarded as belonging to the first or to the second tangent) Consequently Q and Q', P and P' are pairs of corresponding points of the two projective ranges, *i e* the points of contact of the two fixed tangents correspond respectively to the point of intersection of the tangents

Imagine the circle to be generated, as an envelope, by the motion of the variable tangent, the points A, A' will trace out the two projective ranges As the variable tangent approaches the position PQ, the point A' approaches Q', and A approaches the point which corresponds to Q', viz Q, and in the limit when the variable tangent is indefinitely near to PQ, the point A will be indefinitely near to Q or the point of contact of the tangent PQ The point of contact of a tangent must therefore be regarded as the point of intersection of the tangent with an indefinitely near tangent

147 The preceding proposition shows that four tangents a, b, c, d to a circle are cut by a fifth in four points A, B, C, D whose anharmonic ratio is constant whatever be the position of the fifth tangent

This tangent may be taken indefinitely near to one of the four fixed tangents, to a for example, in this case A will be the point of contact of a, and B, C, D the points of intersection ab, ac, ad respectively

As a particular case, if a, b, c, d meet the tangent PQ in four harmonic points, they will meet every tangent in four harmonic points The group constituted by the point of contact of a and the points of intersection ab, ac, ad will also be harmonic In this case, *the four tangents* a, b, c, d *are said to be harmonic* †

* PONCELET, *Propr proj*, Art 462

† STEINER, *loc cit*, p 157, § 43, Collected Works, vol i p 345

148 *The range determined upon any given tangent to a any number of fixed tangents is projective with the pencil joining their points of contact to any arbitrary point on the*

Let $A, B, C,\ \ X$ (Fig 91) be points on the ci $a, b, c,\ \ x$ the tangents at these points respectively points $A', B', C',$ in which the tangent x is cut by the tangents $a, b, c,$ be joined to the centre of the circle, the joining lines will be perpendicular respectively to the chords $XA, XB, XC,$ and will therefore (Art 108) form a pencil equal to the pencil $X(A, B, C,\ \ )$ The range is therefore projective with the pencil $X(A, B, C,\ \ )$

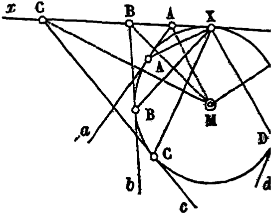


Fig 91

COROLLARY *If four points on a circle are harmonic tangents also at these points are harmonic, and conversely*

For if, in what precedes, $X(ABCD)$ is a harmon $A'B'C'D'$ will be a harmonic range, and conversely

CHAPTER XIV

PROJECTIVE FORMS IN RELATION TO THE CONIC SECTIONS

149 LET the figures be constructed which are homological with those of Arts 144, 146, 148 To the points and tangents of the circle will correspond the points and tangents of a conic section (Art 23) A tangent to a conic is therefore a straight line which meets the curve in two points which are indefinitely near to one another, a point on the curve is the point of intersection of two tangents which are indefinitely near to one another To two equal and therefore projective pencils will correspond two projective pencils, and to two projective ranges will correspond two projective ranges, for two pencils or ranges which correspond to one another in two homological figures are in perspective We deduce therefore the following propositions

(1) *If any number of points $A, B, C, D,$ on a conic are joined to two fixed points O and O' lying on the same conic* (Fig 92), *the pencils $O(A, B, C, D, \quad)$ and $O'(A, B, C, D, \quad)$ so formed are projective with one another To the ray OO' of the first pencil corresponds the tangent at O', and to the ray $O'O$ of the second pencil corresponds the tangent at O*

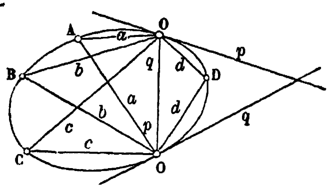


Fig 92

(2) *Any number of tangents $a, b, c, d,$ to a conic determine on a pair of fixed tangents o and o'* (Fig 93) *two projective ranges To the point oo' or Q of the first range corresponds the point of contact Q' of o', and to the same point $o'o$ or P' of the second range corresponds the point of contact P of o* *

* STEINER, *loc cit*, p 139, § 38, Collected Works vol 1 pp 332, 333

(3) *The range which a variable tangent to a conic determin a fixed tangent is projective with the pencil formed by join*

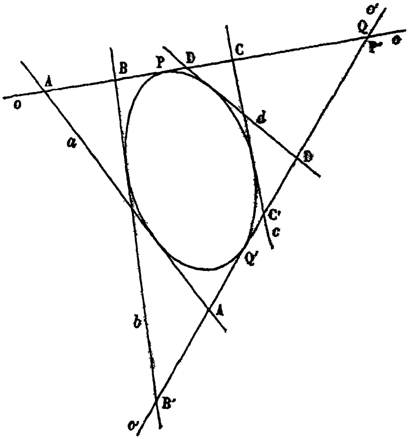


Fig 93

point of contact of the variable tangent to any fixed point conic (Fig 94)

150 We proceed now to the theorems con of Art 149 The proofs here given are due to M Ed Dewulf

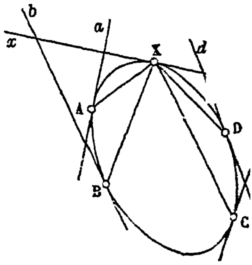


Fig 94.

I *If two (non-concentric) pencils lying in the same plane are projective with one another (but not in perspective), the locus of the points of intersection of pairs of corresponding rays is a conic passing through the centres of the two pencils, and the tangents to the locus at these points are the rays which correspond in t pencils respectively to the straight line which joins the two cen*

Let O and A (Fig 95) be the respective centres of tl pencils, and let OM_1 and AM_1, OM_2 and AM_2, OM_3 and A be pairs of corresponding rays The locus of the poin M_2, M_3, will pass through O, since this point is the section of the ray AO of the pencil A with the correspo ray of the pencil O Similarly A will be a point o locus

tangent to the envelope But this construction is precisely the same as that made use of in Art 23 (Fig 12) in order to draw the curve homological with a circle, taking a given tangent to the circle as axis of homology, any given point O as centre of homology, and s', s'' as a pair of corresponding straight lines The envelope of the lines MM' is therefore a conic section

The theorems (I) and (II) of the present Article are correlative (Art 33), since the figure formed by the points of intersection of corresponding rays of two projective pencils is correlative to that formed by the straight lines joining corresponding points of two projective ranges Thus *in two figures which are correlative to one another (according to the law of duality in a plane), to points lying on a conic in one correspond tangents to a conic in the other*

151. Having regard to Arts 73 and 79, the propositions of Arts. 149, 150 may be enunciated as follows

The anharmonic ratio of the four straight lines which connect four fixed points on a conic with a variable point on the same is constant

The anharmonic ratio of the four points in which four fixed tangents to a conic are cut by a variable tangent to the same is constant *

The anharmonic ratio of four points A, B, C, D lying on a conic is the anharmonic ratio of the pencil $O(A, B, C, D)$ formed by joining them to any point O on the conic *The anharmonic ratio of four tangents a, b, c, d to a conic* is that of the four points $o(a, b, c, d)$, where o is an arbitrary tangent to the conic

If this anharmonic ratio is equal to -1, the group of four points or tangents is termed *harmonic*

The anharmonic ratio of four tangents to a conic is equal to that of their points of contact †

Consequently the tangents at four harmonic points are harmonic, and *vice versa*

The locus of a point such that the rays joining it to four given points $ABCD$ form a pencil having a given anharmonic ratio is a conic passing through the given points

* STEINER, *loc cit*, p 156, § 43, Collected Works, vol 1 p 344

† CHASLES, *Géométrie Supér*, Art 662

The tangent to the locus at one of these points, at A for example, is the straight line which forms with AB, AC, A… a pencil whose anharmonic ratio is equal to the given one.

The curve enveloped by a straight line which is cut by four given straight lines in four points whose anharmonic ratio is given is a conic touching the given straight lines.

The point of contact of one of these straight lines, a for example, forms with the points ab, ac, ad a range whose anharmonic ratio is equal to the given one *

152 *Through five given points O, O', A, B, C in a plane* (Fig 92), *no three of which lie in a straight line, a conic can be described.* For we have only to construct the two projective pencils which have their centres at two of the given points, O and O' for example, and in which three pairs of corresponding rays OA and $O'A$, OB and $O'B$, OC and $O'C$ intersect in the three other points Any other pair OD and $O'D$ of corresponding rays will give a new point D of the curve

To construct the tangent at any one of the given points, at O for example, we have only to determine that ray of the pencil O which corresponds to the ray $O'O$ of the pencil O'

Through five given points *only one* conic can be drawn, for if there could be two such, they would have an infinite number of other points in common (the intersections of all the pairs of corresponding rays of the projective pencils), which is impossible

Given five straight lines o, o', a, b, c in a plane (Fig… *no three of which meet in a* p… *a conic can be described to them.* For we have only to construct the two projective r… which are determined upon t… the given lines, o and o' fo… ample, by the three others a… and of which three pairs of… responding points oa and o… and $o'b$, oc and $o'c$ are… The straight line d whic… any other pair of corresp… points of the two ranges wi… new tangent to the curve

To construct the point of… tact of any one of the… straight lines, that of o f… ample, we have only to dete… that point of the range o… corresponds to the point $o'o$… range o'

Only one conic can be dra… touch five given straight… for if there could be two… they would have an infinite… ber of common tangents (a… straight lines which join p… corresponding points of the… jective ranges), which i… possible

* STEINER, *loc cit*, pp 156, 157, § 43 Collected Works, vol 1 pp 344

From this we see also that

Through four given points can be drawn an infinite number of conics, and two such conics have no common points beyond these four

There can be drawn an infinite number of conics to touch four given straight lines, and two such conics have no common tangents beyond these four

153 The theorems of Art 88 may now be enunciated in the following manner

If a hexagon ab′ca′bc′ is circumscribed to a conic (Figs. 97 and 61), *the straight lines p, q, r which join the three pairs of opposite vertices are concurrent*

If a hexagon AB′CA′BC′ is inscribed in a conic (Figs 98 and 60), *the three pairs of opposite sides intersect one another in three collinear points P, Q, R*

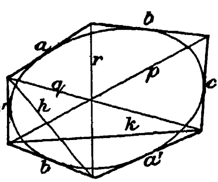


Fig 97

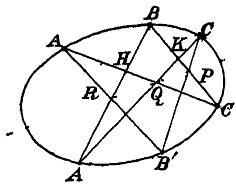


Fig 98

This is known as BRIANCHON'S theorem *

This is known as PASCAL'S theorem †

These results are of such importance in the theory of conics that they deserve independent proofs

The ranges $a(ba'b'c')$ and $c(ba'b'c')$ are projective (Art 149), the pencils formed by joining them to the points (ba'), (bc') respectively are therefore projective If the line joining (ac'), $(a'b)$ be denoted by h, and that joining (bc'), $(a'c)$ by k, the pencils in

The pencils $A(BA'B'C')$ and $C(BA'B'C')$ are projective (Art 149), the ranges in which they cut BA', BC' respectively are therefore projective If AC', $A'B$ cut in H and BC', $A'C$ in K, ranges in question are $(BA'RH)$ and $(BKPC')$ Since they have

* This theorem was published for the first time by BRIANCHON in 1806, and repeated in his *Mémoire sur les lignes du second ordre* (Paris 1817 p 34)

† This theorem was given in PASCAL'S *Essai sur les Coniques*, a small work of six pages 8vo, published in 1640, when its author was only sixteen years old It was republished in the *Œuvres de Pascal* (The Hague, 1779) and again by H WEISSENBORN, in the preface to his book *Die Projection in der Ebene* (Berlin, 1862)

question are $(ba'rh)$ and $(bkpc')$. Since they have the ray b in common, they are in perspective, therefore $(a'k)$, (rp), (hc') are collinear, that is p, q, r are concurrent

the point B in common, th in perspective, therefore A' HC' are concurrent, that is R are collinear

154 Pascal's theorem has reference to six points of a c Brianchon's theorem to six tangents, these six points or gents may be chosen arbitrarily from among all the poin the curve and all the tangents to it. Now a conic is d mined by five points or five tangents, in other words points or five tangents may be chosen at will from amon the points or lines of the plane, but as soon as these elements have been fixed, the conic is determined. Pa theorem then expresses the condition which six points plane must satisfy if they lie on a conic, and Brianc theorem expresses similarly the condition which six str lines lying in a plane must satisfy if they are all tange a conic And the condition in each case is both nece and sufficient

That it is necessary is seen from the theorems th For six points on a conic, taken in any order, ma garded as the vertices of an inscribed hexagon*, bu Pascal's theorem is true for every inscribed hexagon, th pairs of opposite sides must meet in three collinear poi whatever order the six points be taken

The condition is also sufficient For suppose (Fig 98 the hexagon $AB'CA'BC'$, formed by taking the six point certain order, possesses the property that the pairs of op sides BC' and $B'C$, CA' and $C'A$, AB' and $A'B$ intersect in collinear points P, Q, R Through the five points AB' one conic (and one only) can be drawn, if X be the where this conic cuts AC' again, then $AB'CA'BX$ is a scribed hexagon, and its pairs of opposite sides $B'C$ and XA (or $C'A$) and CA', $A'B$ and AB' will meet in three coll points But the second and third of these points are Q

* It is perhaps hardly necessary to remind the reader that the hexag which Pascal's and Brianchon's theorems refer are not hexagons in Euclid —i e they are not necessarily convex (non reentrant) figures

R, therefore BX must meet $B'C$ at the point of intersection of $B'C$ and QR, i e at P Both BC' and BX thus pass through P, and they must therefore coincide Since then the point X lies not only on AC' but also on BC', it must coincide with the point C' itself

The condition is therefore sufficient, and it has already been shown to be necessary

By taking the six points in all the different orders possible, sixty* simple hexagons can be made From the reasoning above, it follows that if any one of these hexagons possesses the property that its three pairs of opposite sides intersect in three collinear points, the six points will lie on a conic, and consequently all the other hexagons will possess the same property †

By analogous considerations having reference to Brianchon's theorem, properties correlative to those just established may be shown to be true of a system of six straight lines ‡

155 Consider the two triangles which are formed, one by the first, third, and fifth sides, the other by the second, fourth, and sixth sides, of the inscribed hexagon $AB'CA'BC'$ (Fig 98) Let BC' and $B'C$, CA' and $C'A$, AB' and $A'B$ be taken as corresponding sides of the triangles By Pascal's theorem these sides intersect in pairs in three collinear points, and therefore (Art 17) the two triangles are homological Pascal's theorem may therefore be enunciated as follows

If two triangles are in homology, the points of intersection of the sides of the one with the non-corresponding sides of the other lie on a conic

Similarly, in a circumscribed hexagon $ab'ca'bc'$ (Fig 97) let the vertices of even order and those of odd order respectively be regarded as the angular points of two triangles, and let bc' and $b'c$, ca' and $c'a$, ab' and $a'b$ be taken to be corresponding vertices By Brianchon's theorem these vertices lie two and two on three straight lines which meet in a point, therefore

* In general, a complete n gon includes in itself $\frac{1}{2}(n-1)(n-2)$ 1 simple n gons

† STEINER *loc cit*, p 311, § 60, No 54, Collected Works, vol 1 p 450

‡ A system of six points on a conic thus determines sixty different lines such as PQR in Fig 98, or *Pascal lines* as they have been called So too a system of six tangents to a conic determines sixty different *Brianchon points*

(Art 16) the two triangles are homological. Brianc[hon's] theorem may therefore be enunciated as follows ·

If two triangles are in homology, the straight lines join[ing the] angular points of the one to the non-corresponding angular [points] of the other all touch a conic

The two theorems may be included under the one enuncia[tion]

If two triangles are in homology, the points of intersection [of the] sides of the one with the non-corresponding sides of the other [lie on] a conic, and the straight lines joining the angular points of t[he one] to the non-corresponding angular points of the other all touch a conic *

156 Returning to Fig 98, let the points A, B', C, A', [be] regarded as fixed, and C' as variable, Pascal's theorem [may] then be presented in the following form

If a triangle $C'PQ$ move in such a way that its sides PQ[, QC',] $C'P$ turn round three fixed points R, A, B respectively, wh[ile two] of its vertices P, Q slide along two fixed straight lines C[B', CA'] respectively, then the remaining vertex C' will describe a conic [which] passes through the following five points, viz the two given [points A] and B, the point of intersection C of the given straigh[t lines, the] point of intersection B' of the straight lines AR and CB', [and the] point of intersection A' of the straight lines BR and CA' †

So also Brianchon's theorem may be expressed [in the] following form

If a triangle $c'pq$ (Fig 99) *move in such a way that its [sides] pq, qc', $c'p$ slide along three fixed straight lines r, a, b respe[ctively,] while two of its sides p, q turn round two fixed points cb', ca' respectively, then the remaining side c' will envelope a conic which touches the following five straight lines, viz the two given straight lines a and b, the straight line c which joins the fixed points, the straight line b' which joins the points ar and cb', and the straight line a' which joins the points br and ca'*

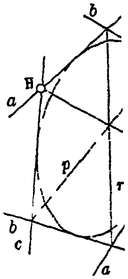


Fig 99

157 (1) If in the theorems of Art 152 (right) one of the tangents is supposed to lie at infinity, the conic becomes a parabola (Art 23) Thus *a parabola is determined by four ta[ngents]*

* MÖBIUS, *loc cit*, Art 278

† This theorem was given by MACLAURIN, in 1721, cf *Phil Trans of t[he]*

or (Art 152, right) *only one parabola can be drawn to touch four given straight lines, and no two parallel tangents can be drawn to a parabola*

(2) If the same supposition is made in theorem (2) of Art 149, it is seen that the points at infinity on the two tangents o and o' are corresponding points of the projective ranges determined on these tangents, for the straight line which joins them is a tangent to the curve It follows (Art 100) that

The tangents to a parabola meet two fixed tangents to the same in points forming two similar ranges, or

Two fixed tangents to a parabola are cut proportionally by the other tangents *.

(3) Let A and A', B and B', C and C', be the points in which the various tangents to the parabola meet the two fixed tangents (Fig 100), and let P and Q' be the respective points of contact of the latter The point of intersection of

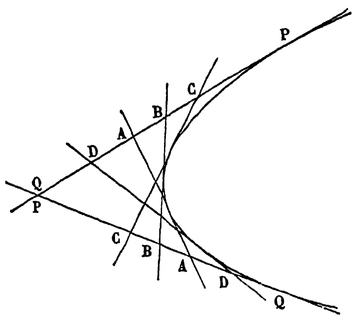


Fig 100

the two fixed tangents will be denoted by Q or P' according as it is regarded as a point of the first or of the second tangent We have then

$$\frac{AB}{A'B'}=\frac{AC}{A'C'}= \quad =\frac{BC}{B'C'}= \quad =\frac{AP}{A'P'}=\frac{AQ}{A'Q'}= \quad =\frac{PQ}{P'Q'}$$

(4) Conversely, *given two straight lines in a plane, on which lie two similar ranges (which are not in perspective), the straight lines connecting pairs of corresponding points will envelope a parabola which*

Society of London for 1735, and CHASLES, *Aperçu historique sur l'origine et le développement des méthodes en Géométrie* (Brussels, 1837, second edition, Paris 1875) If B lies at infinity, the theorem becomes identical with lemma 20 book I of NEWTON'S *Principia*

* APOLLONII PERGAEI *Conicorum* lib III 41

touches the given straight lines at the points which correspond the two ranges respectively to their point of intersection

For the points at infinity on the given straight lines be corresponding points (Art. 99), the straight line which jo them will be a tangent to the envelope, thus the envelop a conic (Art 150 (II)) which has the line at infinity fo tangent, i e it is a parabola.

158 In theorem I of Art 150 (Fig 95) suppose that point A lies at infinity, or, in other words, that the penci consists of parallel rays. To the straight line OA, considerec a ray a' of the pencil O (viz that ray which is parallel to rays of the other pencil), corresponds that ray a of the pe A which is the tangent at the point A This ray a may b a finite, or it may be at an infinite distance

In the first case (Fig 101) the straight line at infini a ray j of the pencil A, and to it corresponds in the penc a ray j' different from a' and consequently not passing thro A, the conic will therefore be a hyperbola (Art 23) ha $A (\equiv aa')$ and jj' for its points at infinity, the straight l is one asymptote and j' is parallel to the other

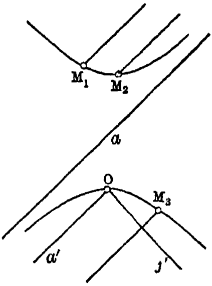


Fig 101

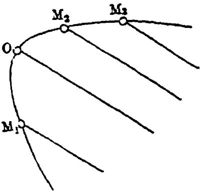


Fig 102

In the second case (Fig 102) the line at infinity i tangent at A to the conic, which is therefore a parabola

159 If in this same theorem of Art 150 the points A O are supposed both to lie at infinity (Fig 103), the two jective pencils will each consist of parallel rays, and the conic which these pencils generate must pass throu and O it is a hyperbola (Art 23) The asymptotes o hyperbola are the tangents to the curve at its infinitely di points *, they will therefore be the rays a and o' of the

* DESARGUES, *loc cit*, p 210, NEWTON *Principia* lib 1 prop 27 Sch

and second pencil which correspond to the straight line at infinity considered as a ray of the second and first pencil respectively.

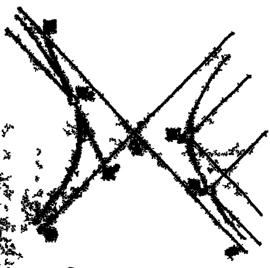


Fig. 109.

By the general theorem of Art. 149, the asymptotes of a hyperbola are cut by the other tangents in points forming two projective ranges, in which the points of contact (which are in this case at infinity) correspond respectively to the point of intersection Q of the asymptotes. The equation of Arts. 74 and 109 (1), viz.

$$JM \cdot I'M' = \text{constant}$$

becomes therefore in this case

$$QM \cdot QM' = \text{constant},$$

M and M' being the points of intersection of any tangent with the asymptotes. We conclude therefore that

The segments which are determined by any tangent to a hyperbola on the two asymptotes (measured from the point of intersection of the asymptotes), are such that the rectangle contained by them is constant

This may be stated in a different form as follows

*The triangle formed by any tangent to a hyperbola and the asymptotes has a constant area**

160 Again, let the theorem of Art 149 be applied to the case of two fixed parallel tangents which are cut by a variable tangent in M and M' In the projective ranges thus generated the points which correspond respectively to the infinitely distant point of intersection of the two fixed tangents are their points of contact, if these be denoted by J and I', we have by Art 74 the equation

$$JM \cdot I'M' = \text{constant}$$

Therefore, *the segments which a variable tangent to a conic cuts off from two fixed parallel tangents (measured from the points of contact of these latter) are such that the rectangle contained by them is constant* †

* APOLLONIUS *loc cit*, III 43

† Ibid III 42

CHAPTER XV

CONSTRUCTIONS AND EXERCISES

161 By help of Pascal's and Brianchon's theorems may solved the following problems

Given five tangents a, b', c, a', b, to a conic, to draw from any given point H, lying on one of these tangents a, another tangent to the curve (Fig 104)

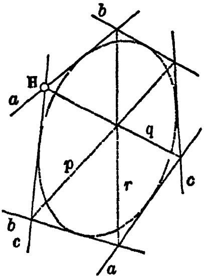


Fig 104

If c' be the required tangent, $ab'ca'bc'$ is a hexagon to which Brianchon's theorem applies Let r be the diagonal connecting one pair ab' and $a'b$ of opposite vertices, and let q be the diagonal connecting another such pair ca' and $c'a$ (where $c'a$ is the given point H), then the diagonal which connects the remaining pair bc' and $b'c$ must pass through the

Given five points A, B', C, B on a conic, to find the poin intersection of the curve with given straight line r drawn thro one of these points A (Fig 105

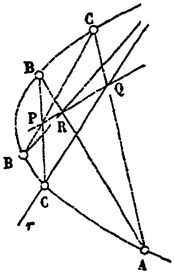


Fig 105

If C' be the required p $AB'CA'BC'$ is a hexagon which Pascal's theorem app Let R be the point of intersec of one pair AB' and $A'B$ of o site sides, and let Q be the p of intersection of another pair CA' and r, then QR n pass through the point of in section of the remaining BC' and $B'C$ If then PE

line joining the points qr and $b'c$, the straight line which joins pb … the re-

… ions … one … epeat- … con- … ber of … may be

… s theorem therefore … struct, by means of its … conic which is deter- … given tangents *

straight line r in the required point C'

By assuming different positions for the given straight line r, all passing through one of the given points on the conic, and repeating in each case the above construction, any desired number of points on the conic may be found

Pascal s theorem therefore serves to construct, by means of its points, the conic which is determined by five given points †

… ular cases of the problem of Art 161 (right)

… the point B to lie at infinity, the problem then … ollowing

… points A, B', C, A' on a hyperbola and the direction … … ote, to find the second point of intersection C' of the … th a given straight line r drawn through A (Fig 106)

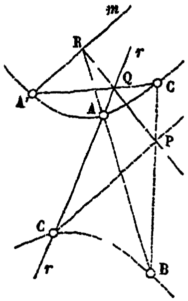


Fig 106

Solution This is deduced from that of the general problem by taking the point B to lie at infinity in the given direction We draw through A' a straight line m in this direction, if then AB' meets m in R, and $A'C$ meets r in Q we join QR meeting $B'C$ in P, and draw through P a parallel to m, this parallel will cut r in the required point C'

II Suppose the point A to lie at infinity, the problem is then

Given four points B', C, A', B on a hyperbola and the direction of one asymptote, to find the point of intersection of the curve with a given straight line r drawn parallel to this asymptote (Fig 107)

Solution Draw through B' a straight line parallel to the given direction If this line meet $A'B$ in R, and if $A'C$ meet r

* BRIANCHON *loc cit*, p 38 PONCELET, *loc cit*, Art 209

† NEWTON, *Principia*, prop 22 MACLAURIN, *De linearum geometricarum proprietatibus generalibus* (London 1748), § 44

in Q, join QR cutting $B'C$ in P Then if BP be joined, will cut r in the required point C'

III Suppose the two points A' and B both to lie at infinit The problem then becomes

Given three points A, B', C on a hyperbola and the directions both asymptotes, to find the second point of intersection of the cur with a given straight line r drawn through A (Fig 108).

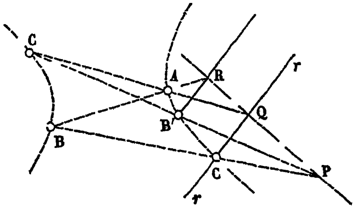


Fig 107

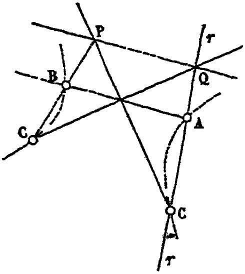


Fig 108

Solution Through the point Q, where the given straig line r meets a straight line drawn through C parallel direction of the first asymptote, draw a parallel to AB P be the point where this parallel cuts $B\ C$, then a parall through P to the second asymptote will cut r in the requir point C'

IV If the two points A and B' both lie at infinity, t problem is

Given three points C, A', B of a hyperbola and the directions both asymptotes, to find the point of intersection of the curve with a given straight line r drawn parallel to one of the asymptotes (Fig 109)

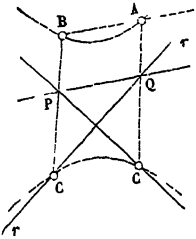


Fig 109

Solution Through Q, the point of intersection of r and CA', draw a parallel to $A'B$, let P be the point where this parallel meets the straight line drawn through C parallel to the other asymptote Then if BP be joined, it will cut r in the required point C'

V If, lastly, the points B', C, A', B are finite and tl straight line AC' lies at infinity, the problem becomes tl

Given four points B', C, A', B of a hyperbola and the direction of one asymptote, to find the direction of the other asymptote (Fig. 110)

Solution Through the point R, in which $A'B$ meets the straight line drawn through B' in the given direction, draw a parallel to CA', let P be the point where this parallel cuts $B'C$ Then if BP be joined, it will be parallel to the required direction.

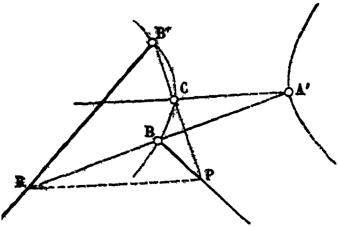


Fig 110

It will be a useful exercise for the student to deduce the constructions for these particular cases from the general construction, in order to do this it is only necessary to remember that to join a finite point to a point lying at infinity in a given direction we merely draw through the former point a parallel to the given direction

163 Particular cases of the problem of Art 161 (left)

I Suppose the point aa' to lie at infinity, then the problem

, a', b *to a conic, to draw the tangent* ...*nem, to* a, *for example* (Fig 111)

...gh the point $a'c$ a straight line q parallel to a, join ab' and $a'b$ by the straight line r, and join the points qr and $b'c$ by the straight line p Then if through the point pb a parallel be drawn to a, it will be the required tangent

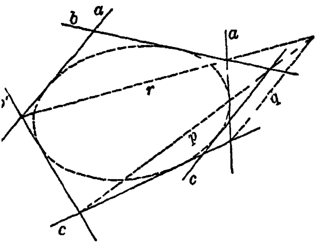


Fig 111

From a given point in the plane of a conic two tangents at most can be drawn to the curve (Art 23), so that from a point lying on a given tangent only one other tangent can be drawn If then the conic is a parabola, it cannot have a pair of parallel tangents (This has already been seen in Art 157 (1))

II Suppose the straight line b to lie at infinity, the problem is then

Given four tangents a, b', c, a' to a parabola, to draw from a giv point H lying on one of them, a, another tangent to the curve (Fi 112)

Solution Through the point ab' draw a straight line r parallel to a', join the points H and $a'c$ by the straight line q, and the points qr and $b'c$ by the straight line p The straight line drawn through H parallel to p will be the required tangent

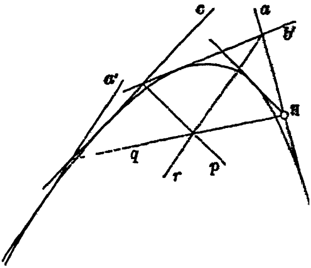


Fig 112

III If the straight line a lies at infinity, we have the problem

Given four tangents b', c, a', b to a parabola, to draw the tange which is parallel to a given straight line (Fig 113)

Solution Through $a'b$ draw the straight line r parallel to b', and through $a'c$ draw the straight line q parallel to the given direction, join the points qr, $b'c$ by the straight line p The straight line through pb parallel to the given direction is the tangent required

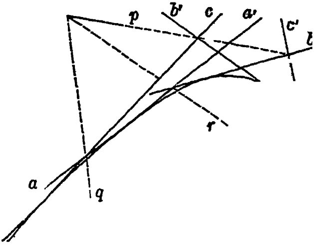


Fig 113

IV If in problem II the point H assume different positio on a, or if in III the given straight line assume differe directions, we arrive at the solution of the problem

To construct by means of its tangents the parabola which is dete mined by four given tangents

CHAPTER XVI

DEDUCTIONS FROM THE THEOREMS OF PASCAL AND BRIANCHON

164. WE have already given some propositions and constructions (Arts. 161–163) which follow immediately from the theorems of Pascal and Brianchon, by supposing some of the elements to pass to infinity Other corollaries may be deduced by assuming two of the six points or six tangents to approach indefinitely near to one another *

If $AB'CA'BC'$ are six points on a conic, Pascal's theorem asserts that the pencils $A(A'B'CC')$ and $B(A'B'CC')$, for example, are projective with one another To the ray AB of the first pencil corresponds in the second the tangent at B, so that we may say that the group of four lines

$$AA', AB', AC, AB$$

is projective with the group

$$BA', BB', BC, \text{tangent at } B$$

But this amounts evidently to saying that the point C', which was at first taken to have any arbitrary position on the curve, has come to be indefinitely near to the point B Instead then of the inscribed hexagon we have now the figure made up of the inscribed pentagon $AB'CA'B$ and the tangent b at the vertex B (Fig 114), and Pascal's theorem becomes the following

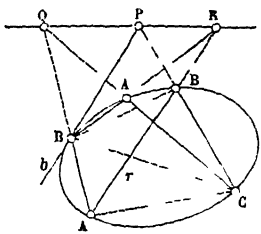


Fig 114

If a pentagon is inscribed in a conic, the points of intersection R, Q of two pairs of non-consecutive sides (AB' and $A'B$, AB and CA'), and the

* CARNOT, *loc cit*, pp 455, 456

point P where the fifth side ($B'C$) meets the tangent at the oppos vertex (B), are collinear

This corollary may also be deduced from the construction (Art. § right) for two projective pencils Three pairs of corresponding ra are here given, viz AA' and BA', AC and BC, AB' and BB' W cut the two pencils by the transversals CA', CB' respectively, if be the point of intersection of $A'B$ and AB', then any pair of corr sponding rays of the two pencils must cut the transversals CA', C respectively in two points which are collinear with R In ord then to obtain that ray of the second pencil which corresponds AB, viz the tangent at B, we join R to the point of intersection Q CA' and AB, and join QR meeting CB' in P, then BP is th required ray b But this construction agrees exactly with the coro lary enunciated above

165 By help of this corollary the two following problems can b solved

(1) *Given five points A, B', C, A', B of a conic, to draw the tange at one of them B* (Fig 114)

Solution Join Q, the point of intersection of AB and CA', to R the point of intersection of AB' and $A'B$, if P is the point wher QR meets $B'C$, then BP will be the required tangent *

Particular cases

Given four points of a hyperbola and the direction of one asym to draw the tangent at one of the given points (This is obt by taking one of the points A, B', C, A' to lie at infinity)

Given four points of a hyperbola and the direction of one asymp to draw that asymptote (B at infinity)

Given three points of a hyperbola and the directions of bot asymptotes, to draw the tangent at one of the given points (Two o the four points A, B', C, A' at infinity)

Given three points of a hyperbola and the directions of both asymptotes, to draw one of the asymptotes (B and one of the othe points at infinity)

(2) *Given four points A B, A', C of a conic and the tangent a one of them B, to construct the conic by points, for example, to find the point of the curve which lies on a given straight line r drawn through A* (Fig 114)

Solution Let R be the point where $A'B$ meets r, and Q the point where AB meets CA', and let QR cut the given tangent in P The point B' where CP cuts the given straight line r will be the one required

By supposing one or more of the elements of the figure to lie at

* MACLAURIN *loc cit* § 40

infinity, *e g* one of the points A, A', C, or two of these points, or the point A and the line r, or the point B, or the point B and one of the other points, or the point B and the given tangent, we obtain the following particular cases

To construct by points a hyperbola, having given

three points of the curve, the tangent at one of these points, and the direction of one asymptote,

or two points, the tangent at one of them, and the directions of both asymptotes,

or three points and an asymptote,

or two points, one asymptote, and the direction of the other asymptote

Given three points of a hyperbola, the tangent at one of them, and the direction of an asymptote, to find the direction of the other asymptote.

To construct by points a parabola, having given three points of the curve (lying at a finite distance) and the direction of the point at infinity on it.

166 Returning to the hexagon $AB'CA'BC'$ inscribed in a conic, let not only C' be taken indefinitely near to B, but also C indefinitely near to B' The figure will then be that of an inscribed quadrangle $AB'A\,B$ together with the tangents at B and B' (Fig 115), and Pascal's theorem becomes the following

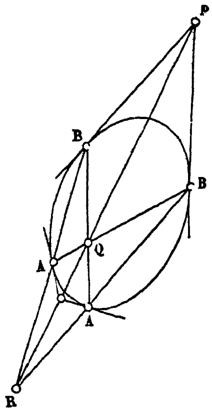


Fig 115

If a quadrangle is inscribed in a conic, the points of intersection of the two pairs of opposite sides, and the point of intersection of the tangents at a pair of opposite vertices, are three collinear points

This property coincides with one already obtained elsewhere (Art 85, right) For considering the projective pencils of which BA and $B'A$, BA' and $B'A'$, are corresponding rays, it is seen that the straight line which joins the point of intersection Q of BA and $B'A'$ to the point of intersection R of $B'A$ and BA' must pass through the point of intersection P of the rays which correspond in the two pencils respectively to the straight line joining their centres B and B'

167 By help of the foregoing corollary the following problems can be solved

(1) *Given four points A, B', A', B of a conic and the tangent BP at one of them B, to draw the tangent at another of the points B'* (Fig 115)

Solution Let AB and $A'B'$ meet in Q, and AB' and $A'B$ in R; and let QR meet the given tangent in P Then $B'P$ will be the required tangent*

By supposing one of the given points, or the given tangent, to lie at infinity, the solutions of the following particular cases are obtained

To draw the tangent at a given point of a parabola, having given in addition two other points on the curve, the tangent at one of them, and the direction of one asymptote, *or*, one other point, the tangent at this, and the directions of both asymptotes, *or*, one other point, one asymptote, and the direction of the other asymptote

To draw the asymptote of a hyperbola when its direction is known, having given in addition three points on the curve and the tangent at one of them, *or*, two points on the curve, the tangent at one of them, and the direction of the second asymptote, *or*, two points on the curve and the second asymptote

To draw the tangent at a given point of a parabola, having given two other finite points on the curve, and the direction of the point at infinity on it

(2) *To construct a conic by points, having given three points A, B, B' on the curve and the tangents BP, $B'P$ at two of them, i e* to determine, for example, the point A' in which an arbitrary straight line r drawn through B is cut by the conic (Fig 116)

Solution Join the point of intersection P of the given tangents to the point R where r cuts AB', and let PR cut AB in Q If $B'Q$ be joined, it will cut r in the required point A'

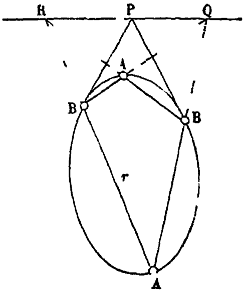


Fig 116

By supposing one of the points A, B, B' or one of the lines BP, $B'P$, r to lie at infinity, we shall obtain the solutions of the following particular cases

To construct by points a hyperbola, having given two points on the curve, the tangents at these, and the direction of one asymptote, *or*, one point on the curve, the tangent there, one asymptote and the direction of the second asymptote, *or*, one point on the curve and both asymptotes

To construct by points a parabola, having given two points on the

* MACLAURIN, *loc cit* § 28

curve, the tangent at one of them, and the direction of the point at infinity on the curve

168 The tangents at the other vertices A and A' of the quadrangle $ABA'B'$ (Fig 116) will also intersect on the straight line joining the points $(AB, A'B')$ and $(AB', A'B)$ Hence the theorem of Art. 166 may be enunciated in the following, its complete form

If a quadrangle is inscribed in a conic, the points of intersection of the two pairs of opposite sides, and the points of intersection of the tangents at the two pairs of opposite vertices, are four collinear points.

If two opposite vertices of the quadrangle be taken to lie at infinity, this becomes the following

If on a chord of a hyperbola, as diagonal, a parallelogram be constructed so as to have its sides parallel to the asymptotes, the other diagonal will pass through the point of intersection of the asymptotes.

169 THEOREM *The complete quadrilateral formed by four tangents to a conic, and the complete quadrangle formed by their four points of contact, have the same diagonal triangle*

In the last two figures write C, D, E, G in place of

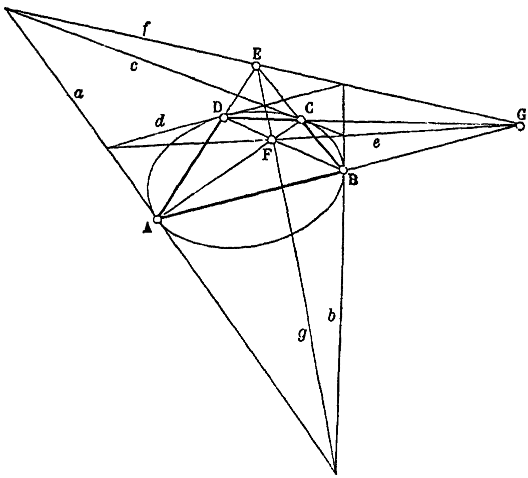


Fig 117

A', B', R, Q respectively In the inscribed quadrangle $ABCD$ (Fig 117) the point of intersection of the tangents at A and C

that of the tangents at B and D, the point of intersection of sides AD, BC, and that of the sides AB, CD all lie on one strai line EG If the same points A, B, C, D are taken in a differ order, two other inscribed quadrangles $ACDB$ and $ACBD$ obtained, to each of which the theorem of Art. 168 may applied Taking the quadrangle $ACDB$, it is seen that point of intersection of the tangents at A and D that of tangents at C and B, the point of intersection of the si AB, CD, and that of the sides AC, BD all lie on one strai line FG So too the quadrangle $ACBD$ gives four poi lying on one straight line EF, viz. the points of intersect of the tangents at A and B, of the tangents at C and D, of sides AD, CB, and of the sides AC, BD*

The three straight lines EG, GF, FE thus obtained are sides of the diagonal triangle EFG (Art 36, [2]) of the comp quadrangle whose vertices are the points A, B, C, D, since the same straight lines contain also the points in wh intersect two and two the tangents a, b, c, d at these poi they are also the diagonals of the complete quadrilat formed by these four tangents The theorem is there proved

170 In the complete quadrilateral $abcd$ the diagona whose extremities are the points ac, bd, cuts the other diagonals g and e in E and G respectively, these two po are therefore harmonically conjugate with regard to ac and (Art 56) The correlative theorem is The two opposite s of the complete quadrangle $ABCD$ which meet in F are h monically conjugate with regard to the straight lines wh connect F with the two other diagonal points E and G (Art Summing up the preceding, we may enunciate the follow proposition (Fig 117)

If at the vertices of a (simple) quadrangle ABCD, inscribed conic, tangents a, b, c, d be drawn, so as to form a (simple) qua lateral circumscribed to the conic, then this quadrilateral possesses following properties with regard to the quadrangle (1) *the diago of the two pass through one point* (F) *and form a harmonic pen* (2) *the points of intersection of the pairs of opposite sides of the lie on one straight line* (EG) *and form a harmonic range,* (3)

* MACLAURIN, *loc cit*, § 50 CARNOT, *loc cit*, pp 453, 454

diagonals of the quadrilateral pass through the points of intersection of the pairs of opposite sides of the quadrangle *

171. By help of the theorem of Art 169, when we are given four tangents a, b, c, d to a conic and the point of contact A of one of them, we can at once find the points of contact of the three others, and when we are given four points A, B, C, D on a conic and the tangent a at one of them, we can draw the tangents at the three other points †

Solution. Draw the diagonal triangle EFG of the complete quadrilateral $abcd$, then AG, AF, AE will cut b, c, d respectively in the required points of contact B, C, D

Draw the diagonal triangle EFG of the complete quadrangle $ABCD$, then the straight lines joining ag, af, ae to B, C, D respectively will be the required tangents

172 The theorem of Art 169 may be enunciated with regard to the (simple) quadrilateral formed by the four straight lines a, b, c, d, it then takes the following form, under which it is seen to be already included in the theorem of Art 170 ‡

In a quadrilateral circumscribed to a conic, the straight lines which join the points of contact of the pairs of opposite sides pass through the point of intersection of the diagonals (Fig 118)

This property coincides with one already proved with regard to two projective ranges (Art 85, left) For consider the projective ranges on a and c as bases, in which ab and cb, ad and cd, are corresponding points, the straight lines which connect the pairs of points ab and cd, cb and ad respectively, must intersect on the straight line which connects the points corresponding in the two ranges respectively to ac, but this is the straight line joining the points of contact of a and c

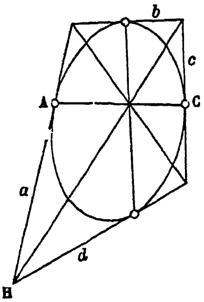


Fig 118

If the conic is a hyperbola, and we consider the quadrilateral which is formed by the asymptotes and any pair of tangents, the foregoing theorem expresses that the diagonals of such a quadrilateral are parallel to the chord which joins the points of contact of the two tangents §

* CHASLES, *Sections coniques*, Art 121

† MACLAURIN, *loc cit*, §§ 38, 39

‡ NEWTON, *loc cit*, Cor ii to lemma XXIV

§ APOLLONIUS, *loc cit*, iii 44

173 The theorem of Art 172 gives the solution of the problem ✓ *To construct a conic by tangents, having given three tangents a, and the points of contact A and C of two of them,* to draw, example, through a given point H lying on a a second tangent to curve (Fig 118)

Solution Join the point ab to the point of intersection of AC $H(bc)$, the joining line will meet c in a point which when joine H gives the required tangent d.

If one of the points A, C or one of the given tangents be suppc to lie at infinity, the solution of the following particular case obtained

To construct by tangents a hyperbola, having given one asympt two tangents to the curve, and the point of contact of one of th *or*, both asymptotes and one tangent

To construct by tangents a parabola, having given th infinity on the curve, two tangents, and the point of cont them, *or*, two tangents and the points of contact of both

Given four tangents to a conic and the point of contac them, to find the points of contact of the others

174 If in Pascal's theorem the points A', B', C' be lie indefinitely near to ~~A, B, C~~ C A B respectively, the figure becomes that of an inscribed triangle ABC together with the tangents at its vertices (Fig 119), and the theorem reduces to the following

In a triangle inscribed in a conic, the tangents at the vertices meet the respectively opposite sides in three collinear points

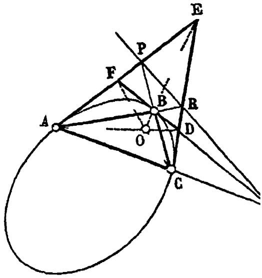


Fig 119

175 This gives the solution of the problem

Given three points A, B, C of a conic and the tangents at tu them A and B, to draw the tangent at the third point C (Fig 119)

Solution Let P, Q be the points where the given tangent A, B cut BC, CA respectively, if PQ cut AB in R, then CR is tangent required

The following are particular cases

Given two points on a hyperbola, the tangents at these po and the direction of one asymptote, to construct the asymp itself

Given one asymptote of a hyperbola, one point on the curve

tangent at this point, and the direction of the second asymptote, to construct this second asymptote

Given both asymptotes of a hyperbola and one point on the curve, to draw the tangent at this point

(From the solution of this problem, it follows that the segment determined on any tangent by the asymptotes is bisected at the point of contact)

Given two points on a parabola, the direction of the point at infinity on the curve, and the tangent at one of the given points, to draw the tangent at the other given point

176 The inscribed triangle ABC and the triangle DEF formed by the tangents (Fig 119) possess the property that their respective sides BC and EF, CA and FD, AB and DE intersect in pairs in three collinear points The triangles are therefore homological, and consequently (Art 18) the straight lines AD, BE, CF which connect their respective vertices pass through one point O Thus we have the proposition

In a triangle circumscribed to a conic, the straight lines which join the vertices to the points of contact of the respectively opposite sides are concurrent

177 By help of this proposition the following problem can be solved

Given three tangents to a conic and the points of contact of two of them, to determine the point of contact of the third

Solution Let DEF (Fig 119) be the triangle formed by the three tangents, and let A, B be the points of contact of EF, FD respectively If AD and BE intersect in O, then FO will cut the tangent DE in the required point of contact C

Particular cases

Given one asymptote of a hyperbola, two tangents, and the point of contact of one of them, to determine the point of contact of the other

Given both asymptotes of a hyperbola, and one tangent, to determine the point of contact of the latter

Given two tangents to a parabola and their points of contact, to determine the direction of the point at infinity on the curve

Given two tangents to a parabola, the point of contact of one of them, and the direction of the point at infinity on the curve, to determine the point of contact of the other given tangent

178 As a particular case of the theorem of Art 176, consider a parabola and the circumscribing triangle formed by the tangents at any two points A, B, and the straight line at infinity, which is also

a tangent If the tangents at A and B meet in C (Fig 120), the straight line joining C to the middle point D of the chord AB is parallel to the direction in which lies the point at infinity on the curve

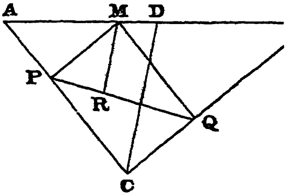


Fig 120.

Again, if any point M be taken on AB, and parallels MP, MQ be drawn to BC, AC respectively to meet AC, BC in P, Q, and if MR be drawn parallel to DC to meet PQ in R, then PQ will be a tangent to the parabola, and R its point of contact.

179 Just as from Pascal's theorem a series of s theorems have been derived, relating to the inscribed tagon, quadrangle, and triangle, so also from Brian theorem can be deduced a series of correlative the relating to the circumscribed pentagon, quadrilateral triangle

Suppose *e g* that two of the six tangents a, b', c, a', b, c' form the circumscribed hexagon (Art 153, left), b and example, lie indefinitely near to one another Since a ta intersects a tangent indefinitely near to it in its point of contact (Arts 146, 149), the hexagon will be replaced by the figure made up of the circumscribed pentagon $ab'ca'b$ together with the point of contact of the side b (Fig 121)

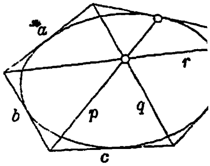


Fig 121

Brianchon's theorem will then become the following

If a pentagon is circumscribed to a conic, the two diagonals connect any two pairs of opposite vertices, and the straight lin ing the fifth vertex to the point of contact of the opposite side in the same point

This theorem expresses a property of projective ranges whic already (Art 85, left) been noticed

For consider the two projective ranges determined by the tangents on a and b as bases Three pairs of corresponding are given, *viz* those determined by a', b', and c Project th range from the point ca' and the second from cb', this give pencils in perspective of which corresponding pairs of rays int

on the straight line r which joins the points ab', ba' In order then to obtain that point of the second range which corresponds to the point ab of the first, *viz* the point of contact of the tangent b, we draw the straight line q which joins the points ca' and ab, and then the straight line p which joins cb' and qr, then pb is the point required. But this construction agrees exactly with the theorem in question.

180 By means of the property of the circumscribed pentagon just established the following problems can be solved

(1). *Given five tangents to a conic, to determine the point of contact of any one of them* *

Particular case Given four tangents to a parabola, to determine their points of contact, and also the direction of the point at infinity on the curve.

(2). *To construct by tangents a conic, having given four tangents and the point of contact of one of them*

Particular cases.

To construct by tangents a hyperbola of which three tangents and one asymptote are given.

To construct by tangents a parabola, having given three tangents and the direction of the point at infinity on the curve, *or* three tangents and the point of contact of one of them

181. The corollaries of Brianchon's theorem which relate to the circumscribed quadrilateral and triangle have already been given (they are the propositions of Arts 172 and 176), they are correlative to the theorems of Arts 166 and 174, just as those of Arts 164 and 179 are correlative to one another

It will be a very useful exercise for the student to solve for himself the problems enunciated in the present chapter the constructions all depend upon two fundamental ones, correlative to one another, and following immediately from Pascal's and Brianchon's theorems

182 The corollaries to the theorems of Pascal and Brianchon show that just as a conic is uniquely determined by five points or five tangents, so also it is uniquely determined by four points and the tangent at one of them, by four tangents and the point of contact of one of them, by three points and the tangents at two of them, or by three tangents and the points of contact of two of them It follows that

(1) An infinite number of conics can be drawn to pass through three given points and to touch a given straight line at one of these points, *or* to pass through two given points and to touch at them two given straight lines, but no two of these conics can have another point in common

* MACLAURIN, *loc cit*, § 41

(2) An infinite number of conics can be drawn to touch a give straight line at a given point, and to touch two other given straigh lines, *or* to touch two given straight lines at two given points, bu no two of these conics can have another tangent in common.

If then two conics touch a given straight line at the same poin (*i.e* if the conics touch one another at this point), they cannot hav in addition more than two common tangents or two common points and if two conics touch two given straight lines at two given point (*i e* if two conics touch one another at two points) they cannot hav any other common point or tangent.

Thus if two conics touch a straight line a at a point A, this poin is equivalent to two points of intersection, and the straight line a i equivalent to two common tangents.

CHAPTER XVII

DESARGUES' THEOREM

183 Theorem. *Any transversal whatever meets a conic and the opposite sides of an inscribed quadrangle in three conjugate pairs of points of an involution.*

This is known as Desargues' 1_ -- *

122) be a
in a conic,

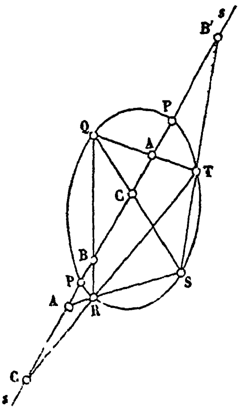


Fig 122

and let s be any transversal cutting the conic in P and P', and

Correlative Theorem *The tangents from an arbitrary point to a conic and the straight lines which join the same point to the opposite vertices of any circumscribed quadrilateral form three conjugate pairs of rays of an involution*

Let $qrst$ (Fig 123) be a quadrilateral circumscribed about a

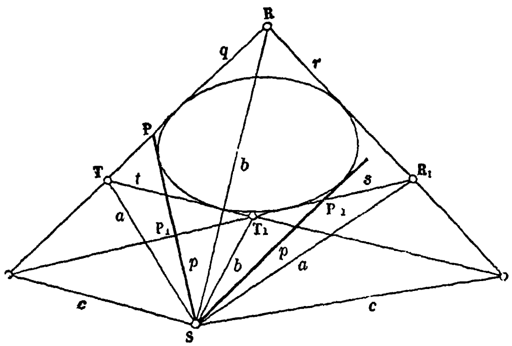


Fig 123

conic, from any point S let tangents p, p' be drawn to the

quadrangle in A, A', B, B' respectively

The two pencils which join the points P, R, P', T of the conic to Q and S respectively are projective with one another (Art 149), and the same is therefore true of the groups of points in which these pencils are cut by the transversal That is, the group of points $PBP'A$ is projective with the group $PA'P'B'$, and therefore (Art 45) with $P'B'PA'$, consequently (Art 123) the three pairs of points

$$PP', AA', BB'$$

are in involution

184 This theorem, like that of Pascal (Art 153, right), enables us to construct by points a conic of which five points P, Q, R, S, T are given For if (Fig 122) an arbitrary transversal s be drawn through P, cutting QT, RS, QR, TS in A, A', B, B' respectively, and if (as in Art 134) the point P' be found, conjugate to P in the involution determined by the pairs of points A, A' and B, B', then will P' be another point on the conic to be constructed

185 The pair of points C, C' in which the transversal cuts the diagonals QS and RT of the inscribed quadrangle belong also (Art 131, left) to the involution determined by the points A, A' and B, B'

Moreover, since the points A A' and B B' suffice to deter

a, a', b, b' be drawn which join S to the vertices qt, rs, qr, ts o the quadrilateral respectively

The two groups of points in which q and s are cut by the tangents p, r, p', t are projective with one another (Art 149), and the same is therefor true of the pencils formed b joining these points to S That is, the group of rays $pbp'a$ projective with the group $pa'p'b$ and therefore (Art. 45) wi $p'b'pa'$, consequently (Art. 12 the three pairs of rays

$$pp', aa', bb'$$

are in involution

This theorem, like that Brianchon (Art 153, left), e ables us to construct by tange a conic of which five tange p, q, r, s, t are given. For (Fig 123) an arbitrary point be taken on p, and this point joined to the points qt, respectively by the rays and if (Art 134) the ray p constructed, conjugate to p in involution determined by the pa of rays a, a' and b, b', then p' be another tangent to the co to be constructed

The pair of rays c, c' wh connect S with the points intersection qs and rt of opposite sides of the circ scribed quadrilateral belong (Art 131, right) to the inv tion determined by the rays a and b, b'

Moreover, since the rays a and b, b' suffice to determine

P, P' are a conjugate pair of this involution for every conic, whatever be its nature, which circumscribes the quadrangle $QRST$

Thus

Any transversal meets the conics circumscribed about a given quadrangle in pairs of points forming an involution.

If the involution has double points, each of these is equivalent to two points of intersection P and P' lying indefinitely near to one another, and will therefore be the point of contact of the transversal with some conic circumscribing the quadrangle

There are therefore either *two* conics which pass through four given points Q, R, S, T and touch a given straight line s (not passing through any of the given points), or there is *no* conic which satisfies these conditions

186 If, from among the six points AA', BB', PP' of an involution, five are given, the sixth is determined (Art 134) If then in Fig 122 it is supposed that the conic is given, and that the quadrangle varies in such a way that the points A, A', B remain fixed, then also the point B' will remain invariable, consequently

If a variable quadrangle move in such a way as to remain always inscribed in a given conic, while three of its sides turn each round one of three fixed collinear points then the fourth side will

conjugate pair of this involution for every conic, whatever be its nature, which is inscribed in the quadrilateral $qrst$

Thus

The pairs of tangents drawn from any point to the conics inscribed in a given quadrilateral form an involution.

If the involution has double rays, each of these is equivalent to two tangents p and p' lying indefinitely near to one another, and will therefore be the tangent at S to some conic inscribed in the quadrilateral

There are therefore either *two* conics which touch four given straight lines q, r, s, t and pass through a given point S (not lying on any of the given lines), or there is *no* conic which satisfies these conditions

If, from among the six rays aa', bb', pp' of an involution, five are given, the sixth is determined (Art 134) If then in Fig 123 it is supposed that the conic is given, and that the quadrilateral varies in such a way that the rays a, a', b remain fixed, then also the ray b' will remain invariable, consequently

If a variable quadrilateral move in such a way as to remain always circumscribed to a given conic, while three of its vertices slide each along one of three fixed concurrent straight lines, then the

collinear with the three given ones	*fourth fixed straight line, concu rent with the three given ones.*

187 The theorem of the preceding Art (left) may be ex tended to the case of any inscribed polygon having an eve number of sides Suppose such a polygon to have $2n$ side and to move in such a way that $2n-1$ of these pass respe tively through as many fixed points all lying on a straigl line s (Fig 124) Draw the diagonals connecting the first of its vertices with the 4^{th}, 6^{th}, 8^{th}, $2(n-1)^{th}$ vertex, thus dividing the polygon into $n-1$ simple quadrangles In the first of these quadrangles the first three sides (which are the first three sides of the polygon) pass respectively throu three fixed points on s, therefore also the fourth side (whicl the first diagonal of the polygon) will pass through a fixed po on s In the second quadrangle the first three sides (the fi diagonal and the fourth and fifth side of the polygon) p respectively through three fixed points on s, therefore fourth side (the second diagonal of the polygon) will p through a fixed point on s Continuing in the same manr we arrive at the last quadrangle and find that the fourth s of this (*i e* the $2n^{th}$ side of the polygon) passes througl fixed point on s We may therefore enunciate the gene theorem

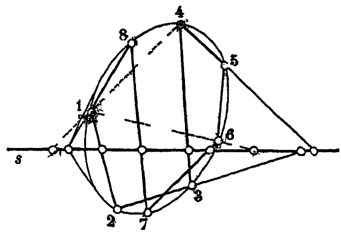


Fig 124.

If a variable polygon of an even number of sides move in suc way as to remain always inscribed in a given conic, while all its s but one pass respectively through as many fixed points lying o straight line, then the last side also will pass through a fixed p collinear with the others *

If tangents can be drawn to the conic from the fixed p round which the last side turns, and if each of these tang is considered as a position of the last side, the two vert which lie on this side will coincide and the polygon will l only $2n-1$ vertices The point of contact of each of the

angents will therefore be one position of one of the vertices f a polygon of $2n-1$ sides inscribed in the conic so that its ides pass respectively through the $2n-1$ given collinear oints.

188 The solution of the correlative theorem is left as an xercise to the student the enunciation is as follows

If a variable polygon of an even number ($2n$) of sides moves so as to emain always circumscribed to a given conic, while all its vertices but one slide along as many fixed straight lines radiating from a centre, then the last vertex also will slide along a fixed straight line passing through the same centre (Fig 125)

If the straight line on which this last vertex slides cut the conic in two points, and if the tangents at these be drawn, each of them will be one position of a side of a polygon of $2n-1$ sides circumscribed about the conic so that its vertices lie each on one of the $2n-1$ given concurrent straight lines

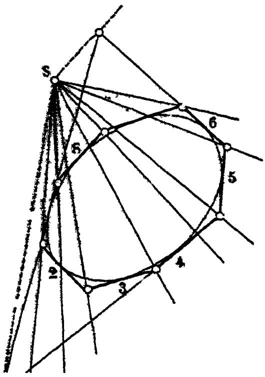


Fig 125

189 If in Fig 122 it be supposed that the points S and T lie ndefinitely near to one another on he conic, or in other words that ST is the tangent at S, then the quadrangle $QRST$ reduces to the nscribed triangle QRS and the angent at S (Fig 126), so that Desargues theorem becomes the ollowing

If a triangle QRS is inscribed n a conic, and if a transversal s neet two of its sides in A and A', he third side and the tangent at he opposite vertex in B and B'

If in Fig 123 the tangents s and t be supposed to lie indefinitely near to one another, so that st becomes the point of contact of the tangent s, then the quadrilateral $qrst$ reduces to the circumscribed triangle qrs and the point of contact of s (Fig 127), so that the theorem correlative to that of Desargues becomes the following

If a triangle qrs is circumscribed about a conic, and if from any point S there be drawn the straight lines a, a' to two of its vertices the straight lines b, b' to

these three pairs of points are in involution

190 This theorem gives a solution of the problem *Given five*

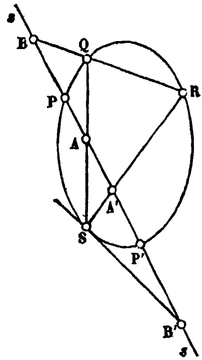


Fig 126

points P, P', Q, R, S on a conic, to draw the tangent at any one of them S

For if A, A', B (Fig 126) are the points in which the straight line PP' cuts the straight lines QS, SR, RQ respectively, we construct (as in Art 134) the point B' conjugate to B in the involution determined by the two pairs of points A, A' and P, P', then $B'S$ will be the required tangent

191 If in Fig 126 it be now supposed in addition that the points Q and R also lie indefinitely near to one another on the conic, *i e* that QR is the tangent at Q, then the inscribed quadrangle $QRST$ is replaced by the two tangents at Q and S and their chord of contact QS counted twice (Fig 128)

contact of the opposite side, and the tangents p, p' to the conic, then these three pairs of rays are in involution

This theorem gives a solution of the problem *Given five tangent*

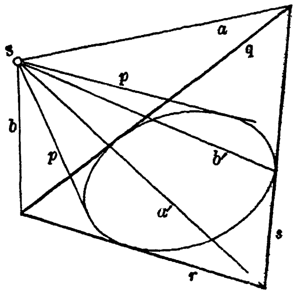


Fig 127

p, p', q, r, s to a conic, to find th point of contact of any one of them s

For if a, a', b (Fig 127) are th rays joining the point pp' to th points qs, sr, rq respectively, w construct (as in Art 134) the ra b' conjugate to b in the involu tion determined by the two pai of rays a, a' and p, p', then b will be the required point of con tact

If in Fig 127 it be now sup posed in addition that the ta gents q and r lie indefinitely ne to one another *i e* that qr is th point of contact of the tangent then the circumscribed quadr lateral $qrst$ is replaced by th points of contact of the tangen q and s and the point of interse tion qs of these tangents count twice (Fig 129)

also coincide in one point, which is consequently one of the double points of the involution determined by the pairs of conjugate

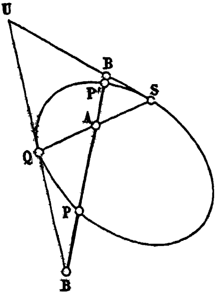


Fig 128

points P, P' and B, B' In this case, then, Desargues' theorem becomes the following

If a transversal cut two tangents to a conic in B and B', their chord of contact in A, and the
in P and P', then the point A is a double point of the involution determined by the pairs of points P, P' and B, B'

Or, differently stated

If a variable conic pass through two given points P and P' and touch two given straight lines, the chord which joins the points of contact of these two straight lines will always pass through a fixed point on PP'

If the tangents QU, SU vary at the same time with the conic, while the points P P' B B' re-

rays a and a' will also coincide in a single ray a, which is consequently one of the double rays of the involution determined by the

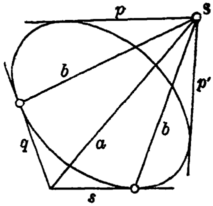


Fig 129

pairs of conjugate rays p, p' and b, b' The theorem correlative to that of Desargues then becomes the following

If a given point S be joined to two points on a conic by the straight lines b, b', and to the point of intersection of the tangents at these points by the straight line a, and if from the same point S there be drawn the two tangents p, p' to the conic, then a is a double ray of the involution determined by the pairs of rays p, p' and b, b'

Or, differently stated

If a variable conic touch two given straight lines p and p' and pass through two given points, the tangents at these two points will always intersect on a straight line passing through pp'

If the points of contact of q and s vary at the same time with the conic while the straight lines

QS must still always pass through one or other of the double points of the involution determined by the pairs of points P, P' and B, B' If then four collinear points P, P', B, B' are given and any conic is drawn through P and P', and then the pairs of tangents from B and B' to this conic, then if each tangent from B is taken together with each tangent from B', four chords of contact will be obtained, which intersect one another two and two in the double points of the involution determined by P, P' and B, B' *

of intersection qs must still always lie on one or other of the double rays of the involution determined by the pairs of rays p, p' and b, b' If then four concurrent straight lines p, p', b, b' are given and any conic is drawn touching p and p', and then the two pairs of tangents to this conic at the points where it is cut by b and b', then if the tangents at the two points on b are combined with the tangents at the two points on b', each with each, four points of intersection will be obtained, which lie two and two on the double rays of the involution determined by p, p' and b, b'

X **192** From the theorem of the last Article (left) is derived a solution of the problem *Given four points P, P', Q, S on a conic and the tangent at one of them Q, to draw the tangent at any other of the given points S* (Fig 128)

For if A, B are the points in which PP' cuts QS and the given tangent respectively, and we construct the point B' conjugate to B in the involution determined by the pair of points P, P' and the double point A, then the straight line SB' will be the tangent required

From the theorem of the last Article (right) is derived a solution of the problem *Given four tangents p, p', q, s to a conic and the point of contact of one of them q, to determine the point of contact of any other of the given tangents s* (Fig 129)

For if a, b are the rays which connect pp' with qs and with the given point of contact respectively, and we construct the ray b' conjugate to b in the involution determined by the pair of rays p, p' and the double ray a then sb' will be the required point of contact

193 Consider again the theorem of Art 191, and suppose that the conic is a hyperbola, and that its asymptotes are the tangents given (Fig 130) The chord of contact QS lies in this case entirely at infinity, so that the involution (PP', BB',) has one double point at infinity, and therefore (Arts 59, 125) the other double point

the common point of bisection of the segments PP', BB'. We onclude that

If a hyperbola and its asymptotes be cut by a transversal, the segments intercepted by the curve and by the asymptotes respectively have the same middle point

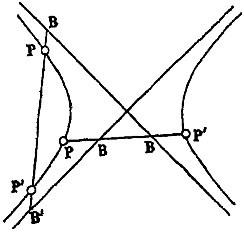


Fig 130

From this it follows that

$$PB = B'P' \text{ and } PB' = BP' \text{*},$$

which gives a rule for the construction of a hyperbola when the two asymptotes and a point on the curve are given †

194. Consider once more the heorem of Art 191 (left), and uppose now that the points P nd P' are indefinitely near to one e let the transversal nt to the conic (Fig its point of contact P will

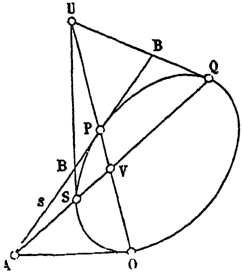


Fig 131

e the second double point of the nvolution determined by the pair f points B, B' and the double oint A, consequently (Art 125) and A are harmonic conjugates

Consider once more the theorem of Art 191 (right), and suppose now that the tangents p and p' lie indefinitely near to one another, *i e* let the point S lie on the conic itself (Fig 132) The tangent to the conic at S will be the

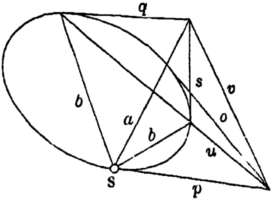


Fig 132

second double ray of the involution determined by the pair of rays b, b' and the double ray a, consequently (Art 125) p and a are harmonic conjugates with

* [illegible]

with regard to B and B', and we conclude that

In a triangle UBB' circumscribed to a conic, any side BB' is divided harmonically by its point of contact P and the point where it meets the chord QS joining the points of contact of the other two sides

195 From A a second tangent can be drawn to the conic, let its point of contact be O Since the four points P, A, B, B', which have been shown to be harmonic, are respectively the point of contact of the tangent AB, and the three points where this tangent cuts three other tangents OA, QB, SB' respectively, it follows that the tangents AB, OA, QB, SB' will be cut by every other tangent in four harmonic points (Art 149), i e they are four harmonic tangents (Art 151) And since the chord of contact QS of the conjugate tangents QB, SB' passes through A the point of intersection of the tangents at P and O, we have the theorem

If the chord of contact of one pair of tangents to a conic pass through the point of intersection of another pair of tangents, then each pair is harmonically conjugate with regard to the other

And conversely

If four tangents to a conic are harmonic, the chord of contact of each pair of conjugate tangents passes through the point of intersection of the other pair

regard to b and b', and we conclude that

In a triangle ubb' inscribed in a conic, any two sides b and b' are harmonic conjugates with regard to the tangent p at the vertex in which they meet and the straight line joining this vertex to the point of intersection of the tangents q and s at the other two vertices.

The straight line a cuts the conic in a second point, let the tangent at this be o Since the four rays p, a, b, b', which have been shown to be harmonic, are respectively the tangent at S, and the straight lines which join S to three other points on the conic (the points of contact of o, q, and s) it follows that the straight lines connecting these four points with any other point on the conic will form a harmonic pencil (149), i e the four points harmonic (Art 151) And . the point of intersection of tangents q and s lies on the chord of contact of the tangents p and o, we have the theorem

If the point of intersection of the tangents at one pair of points on a conic lie on the chord joining another such pair of points, then each pair is harmonically conjugate with regard to the other

And conversely

If four points on a conic are harmonic, the point of intersection of the tangents at each pair of conjugate points lies on the chord joining the other pair

by virtue of the property already established (Arts 148, 149) that the tangents at four harmonic points on a conic are themselves harmonic, and conversely We may then enunciate as follows

If a pair of tangents to a conic meet in a point lying on the chord of contact of another pair, then also the second pair will meet in a point lying on the chord of contact of the first, and the four tangents (and likewise their points of contact) will form a harmonic system *

Thus in Fig. 131 QS passes through A, the point of intersection of PA and OA, and similarly OP passes through U the point of intersection of QB and SB', and the pencil $U(QSPA)$ is harmonic, and likewise the pencil $A(OPQU)$

In Fig 132 the point qs lies on a, the chord of contact of o and p, and similarly the point op lies on the straight line u which joins the points of contact of q and s, and the range $u(qsap)$ is harmonic, and the range $a(opqu)$ also.

197 Example Suppose the conic to be a hyperbola (Fig 133)

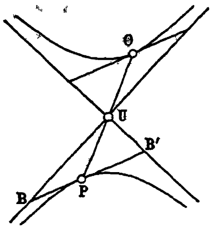


Fig 133

Its asymptotes are a pair of tangents whose chord of contact QS is the straight line at infinity, consequently the chord joining the points of contact of a pair of parallel tangents will pass through the point of intersection U of the asymptotes, and conversely, if through U a transversal be drawn, the tangents at the points P and O, where it cuts the curve, will be parallel The point U will lie midway between P and O, since in general $UVPO$ (Fig 131) is a harmonic range, and in this case V lies at infinity

Any tangent to the curve cuts the asymptotes in two points B and B' which are harmonically conjugate with regard to the point of contact P and the point where the tangent meets the chord of contact of the asymptotes, but this last lies at infinity, therefore P is the middle point of BB' Thus

The part of a tangent to a hyperbola which is intercepted between the asymptotes is bisected at its point of contact †

This proposition is a particular case of that of Art 193

198 THEOREM ‡ *If a quadrangle is inscribed in a conic, the rectangle contained by the distances of any point on the curve from*

* DE LA HIRE, *loc cit*, book 1 prop 30 STEINER *loc cit*, p 159, § 43, Collected Works, vol 1 p 346

† APOLLONIUS *loc cit*, II 319

‡ To this CHASLES has given the name of PAPPUS' theorem, since it corresponds

one pair of opposite sides is to the rectangle contained by its distances from the other pair in a constant ratio

In Fig 122, the pairs of points P and P', A and A', B and B' being, by Desargues' theorem, in involution, the anharmonic ratios $(PP'AB)$ and $(P'PA'B')$ are equal to one another, or

$$\frac{PA}{P'A} \quad \frac{PB}{P'B} = \frac{P'A'}{PA'} \quad \frac{P'B'}{PB'}$$
$$= \frac{PB'}{P'B'} \quad \frac{PA'}{P'A'}$$

But PA $P'A$ is equal to the ratio of the distances (measured in any the same direction) of the points P and P' from the straight line QT, and the other ratios in the foregoing equation may be interpreted similarly, we have therefore

$$\frac{(A)}{(A)'} \quad \frac{(B)}{(B)'} = \frac{(B')}{(B')'} \quad \frac{(A')}{(A')'},$$

or
$$\frac{(A) \quad (A')}{(B) \quad (B')} = \frac{(A)' \quad (A')'}{(B)' \quad (B')'},$$

where (A), (A'), (B), (B') denote the distances of the point P from the sides QT, RS, QR, ST respectively of the inscribed quadrangle $QRST$, and $(A)'$, $(A')'$, $(B)'$, $(B')'$ denote similarly the distances of the point P' from these sides respectively (These distances may be measured either perpendicularly or obliquely, so long as they are measured parallel to one another) The ratio

$$\frac{(A)\,(A')}{(B)\,(B')}$$

is therefore constant for all points P on the conic, which proves the theorem

199 THEOREM *If a quadrilateral is circumscribed about a conic the rectangle contained by the distances of one pair of opposite vertices from any tangent is to the rectangle contained by the distances of the other pair from the same tangent in a constant ratio* *

In Fig 123 let the vertices qr, qt, st, sr of the circumscribed quadrilateral $qrst$ be denoted by R, T, T_1, R_1 respectively, let the points where the tangents p, p' meet the side q be called P, P' respectively †, and let the points where these same tangents meet the side s be called P_1, P_1' respectively Since by the theorem correlative to that of Desargues, the pairs of rays p and p', a and a', b and b', are in involution, the anharmonic ratios $(bapp')$ and $(b'a'p'p)$ are equal to one another Hence by theorem (2) of Art 149,

* CHASLES *Sections coniques* Art 26

$$(RTPP') = (T_1R_1P_1'P_1)$$
$$= (R_1T_1P_1P_1') \text{ by Art. } 45,$$
$$\frac{RP}{TP} \ \frac{RP'}{TP'} = \frac{R_1P_1}{T_1P_1} \ \frac{R_1P_1'}{T_1P_1'},$$

whence
$$\frac{RP \ \ T_1P_1}{TP \ \ R_1P_1} = \frac{RP' \ \ T_1P_1'}{TP' \ \ R_1P_1'}$$

But $RP \quad TP$ is equal to the ratio of the distances (measured in any the same direction) of the points R and T from the straight line p; so also $T_1P_1 \quad R_1P_1$ is the ratio of the distances of the points T_1 and R_1 from the same straight line p The foregoing equation therefore expresses that the ratio

$$\frac{RP \ \ T_1P_1}{TP \ \ R_1P_1}$$

is constant for every tangent p to the conic, which proves the theorem.

CHAPTER XVIII

SELF-CORRESPONDING ELEMENTS AND DOUBLE ELEMENTS.

200 CONSIDER two projective flat pencils, concentric or no concentric Through their common centre or through the two centres O and O' draw a conic or a circle, and let th cut the rays of the first pencil in $A, B, C,$ and those of t second in $A', B', C',$ Project these two series of poin from two new points O_1, O_1' (or from the same point) lyi on the conic, the two projecting pencils $O_1(ABC \quad)$ a $O_1'(A'B'C' \quad ,)$ are by Art 149 projective with the two giv pencils $O(ABC \quad)$ and $O'(A'B'C' \quad)$ respectively, and a therefore projective with one another

*The two series of points ABC and $A'B'C'$ are said t two projective ranges on the conic**

I Now project these two ranges (Fig 134) from two o corresponding points, say from A' and A The projecti pencils

$$A'(A, B, C, \quad) \text{ and } A(A', B', C', \quad)$$

will be projective with one another, and since they have t self-corresponding ray AA', they are in perspective Corresponding pairs of rays will therefore (Art 80) intersect on a fixed straight line, so that AB' and $A'B$, AC' and $A'C$, AD' and $A'D$, will meet on one straight line s If any point be taken on s, the straight lines joining it to A and A' will cut the conic again in another pair of corresponding points the ranges $ABCD$ and $A'B'C'D'$

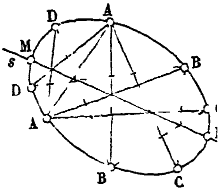


Fig 134

* BELLAVITIS, *Saggio di Geometria derivata* (Nuovi Saggi dell' Accademi

If instead of A' and A any other pair of corresponding points had been taken as centres of projection, say B' and B, the same straight line s would have been arrived at For since $AB'CA'BC'$ is a hexagon inscribed in a conic, it follows by Pascal's theorem that the point of intersection of $B'C$ and BC' must lie on the straight line which joins the point of intersection of $A'B$ and AB' to that of $A'C$ and AC' (Art 153, right).

II. Any point M in which the conic and the straight line s intersect is a *self-corresponding point* of the two ranges ABC and $A'B'C'$ For if M, M' be corresponding points

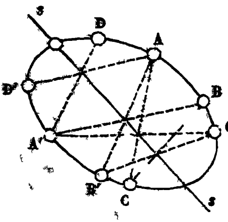


Fig 135

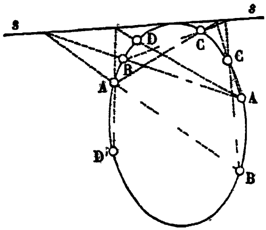


Fig 136

of the two ranges, it has been seen that $A'M$, AM' must intersect on s, if then M lie on s, M' must coincide with M, *i e* a pair of corresponding points of the two ranges are united at M

The two ranges will therefore have two self-corresponding points, or only one, or none at all, according as the straight line s cuts the conic in two points (Fig 135), *touches it* (Fig 136), *or does not cut it* (Fig 137)

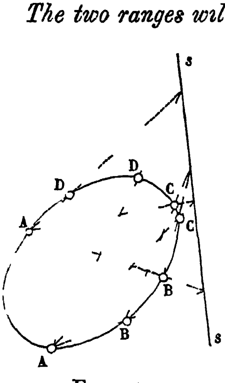


Fig 137

III From what precedes it is clear that *two projective ranges of points on a conic are determined by three pairs of corresponding points A and A', B and B', C and C'* For in order to find other pairs of corresponding points, and the self-corresponding points (when such exist), we have only to construct the straight line s which passes through the points of intersection of the three pairs of opposite sides of the hexagon $AB'CA'BC'$

be the points where s cuts the conic, and any number of pa of corresponding points can be constructed by help of t property that any pair D and D' are such that the lines A and AD' (or $B'D$ and BD', or $C'D$ and CD') intersect on s

201 Instead of *projective ranges of points on a conic* we m consider *projective series of tangents* to the same Let o, o' be t projective ranges of points (either collinear or lying on different strai lines as bases) Describe a conic to touch o and o', and draw to t conic, from each pair of corresponding points A and A', B and C and C', the tangents a and a', b and b', c and c', If n these two series of tangents are cut by two other tangents o_1 and two new ranges of points will be obtained, which are projective w the given ranges respectively (Art. 149), and are therefore project with one another

Two series of tangents to a conic are said to be projective with another when they are cut by any other tangent to the curve in t projective ranges

I Suppose the first series of tangents to be cut by the tangent and the second by the tangent a The two projective ranges formed are in perspective, since they have the self-correspond point aa', the straight lines which join the pairs of correspond points $a'b$ and ab', $a'c$ and ac', will therefore pass through point S This point does not change if another pair of tange b' and b are taken as transversals, for by Brianchon s theorem straight lines which join the three pairs of opposite vertices $a'b$ ab', $a'c$ and ac', $b'c$ and bc' of the circumscribed hexagon $ab'c$ a must meet in a point (Art 153, left)

II If the point S is such that tangents can be drawn from it the conic, each of them will be a *self corresponding line* of the t projective series of tangents abc and $a'b'c'$

[The proof of this is analogous to that of the corresponding prope of two projective ranges of points on a conic (Art 200, II)]

III Two projective series of tangents to a conic are determi by three pairs of corresponding lines a and a', b and b', c and For in order to find other pairs of corresponding lines, and the s corresponding lines (when such exist), we have only to construct point of intersection S of the diagonals which join two and two opposite vertices of the circumscribed hexagon $ab'c\,a'bc'$ The s corresponding lines will be the tangents from S to the conic, and a pair of corresponding lines d and d' may be constructed by means the property that the points $a'd$ and ad' (or $b'd$ and bd', or $c'd$ a cd') are collinear with S

IV A range of points $A, B, C,$ on a conic and a series of tangents $b, c,$ to the same are said to be projective with one another, hen the pencil formed by joining $A, B, C,$ to any point on the nic is projective with the range determined by $a, b, c,$ on any ngent to the conic

A range of points $A, B, C,$ on a conic, or a series of tangents $b, c,$ to the same, is said to be projective with a range of points a straight line, or a pencil (flat or axial), when this last-mentioned nge or pencil is projective with the pencil formed by joining BC to any point on the conic or with the range determined by $b, c,$ on any tangent to the conic

V These definitions premised, we may now include under the le of *one-dimensional geometric form* not only the range of linear points, the flat pencil, and the axial pencil, but also e range of points on a conic and the series of tangents to a nic*, and with regard to these we may enunciate the general eorem *Two one-dimensional forms which are each projective ith a third (also of one dimension) are projective with one another* f. Art. 41)

VI From these definitions it follows also that theorem (3) of rt. 149 may be enunciated in the following manner

Any series of tangents to a conic is projective with the range formed their points of contact

VII Let $A, B, C,$ and $A', B', C',$ be two projective ranges of oints on a conic, and let $a, b, c,$ and $a', b', c',$ be the tangents these points The series of tangents $a, b, c,$ and $a', b', c',$ e projective with the series of points of contact $A, B, C,$ and $', B', C',$ respectively, and are therefore projective with one nother Let s be the straight line on which the pairs of straight lines ich as AB' and $A'B$, AC' and $A'C$, BC' and $B'C$ intersect, and t S be the point in which meet the straight lines joining pairs of oints such as ab' and $a'b$, ac' and $a'c$, bc' and $b'c$, If s cuts the nic in two points M and N, these must be the self-corresponding oints of the ranges ABC and $A'B'C'$, the tangents m and n M and N respectively must therefore be the self corresponding nes of the projective series abc and $a'b'c'$, consequently the raight lines m and n will meet in S

VIII From the foregoing it follows that for the consideration of a

* The introduction of these new one dimensional forms enables us now to add the operations previously made use of (section by a transversal straight line d projection by straight lines radiating from a point) two others, *viz* section of lat pencil by a conic passing through the centre of the pencil, and projection of ange of collinear points by means of the tangents to a conic which touches the

series of tangents can always be substituted that of their points of contact, and *vice versa*

202 Instead of considering any two projective pencils as in Art 200, take an involution of straight lines radiating from a point O Suppose these to be cut by a conic passing through O in the pairs of points A and A', B and B', C and C', , and let these points be joined to any other point O_1 on the conic Since by hypothesis (Arts 122, 123) the pencils $O(AA'BC\ \ )$ and $O(A'AB'C'\ \ )$ are projective with one another, the pencils $O_1(AA'BC\ \ )$ and $O_1(A'AB'C'\ \ )$ are so too (Art 149), and therefore the rays issuing from O_1 form an involution also In this case we say that *the two projective ranges of points ABC and $A'B'C'$ on the conic form an involution*, or that *there is on the conic an involution formed by the pairs of conjugate points AA', BB', CC',* *

I Similarly, if there is given an involution of points on a straight line o and if from the pairs of conjugate points there be drawn tangents a and a', b and b', c and c', to a conic touching o, these will be cut by any other tangent to the conic in an involution of points, in this case we say that aa', bb', cc', form *an involution of tangents to the conic* (cf Art 201)

II If several pairs of tangents aa', bb', cc' to a conic form an involution, their points of contact AA', BB', CC', form an involution also, and conversely (Art 201, VI)

203 Of the six points A, B', C, A', B, C' on a conic considered in Art 200, let C' lie indefinitely near to A, and C indefinitely near to A' The projective ranges $(ABC\ \ )$ or $(ABA'\ \ )$ and $(A'B'C'\ \ )$ or $(A'B'A\ \ )$ will then form an involution $(AA', BB',\ \ )$ and the inscribed hexagon is replaced by the figure made up of the inscribed quadrangle $AB'A'B$ and the tangents at the opposite vertices A and A' (Figs 115, 138) We conclude that

An involution of points on a conic is determined by two pairs AA', BB'

I In order to find other pairs of conjugate points it is only necessary to construct the straight line s which joins the point of intersection of AB' and $A'B$ to that of AB and $A'B'$, i e to

* STAUDT *Beiträge zur Geometrie der Lage* (Nürnberg, 1856-57-60), Arts 70

draw the straight line joining the points of intersection of the pairs of opposite sides of the inscribed quadrangle $AB'A'B$ The points where s cuts the conic are the double points Pairs of conjugate points will be constructed by remembering that any pair C and C' are such that the straight lines AC and $A'C'$ (or AC' and $A'C$, or BC and $B'C'$, or $B'C$ and BC') intersect on s

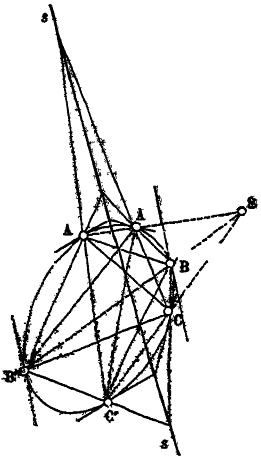


Fig. 138

II. The tangents at a pair of conjugate points, such as A and A', B and B', .. likewise intersect on the straight line s (Art 166)

III. Since the pairs of sides BC and $B'C'$, CA and $C'A'$, AB and $A'B'$ of the triangles ABC, $A'B'C'$ intersect in three points lying on a straight line s, the triangles are homological (Art. 17)*, and the straight lines AA', BB', CC' will meet in one point S But AA' and BB' suffice to determine this point, accordingly

Any pair of conjugate points of the involution are collinear with a fixed point S, or

Every straight line drawn through S to cut the conic determines on it a pair of conjugate points of the involution

IV It has been seen that if s cuts the conic in two points M and N, these are the double points of the involution The tangents at M and N will therefore meet in S

V Conversely, *the pairs of points in which a conic is cut by the rays of a pencil whose centre S does not lie on the curve form an involution*

For if A and A', B and B' are the points of intersection of the curve with two of the rays, these two pairs AA' and BB' determine an involution such that the straight line joining any pair of corresponding points always passes through a fixed point, viz S If the involution has double points, these are the intersections of the conic with the

* The triangles $A'BC$ and $AB\,C$, $AB'C$ and $A\,BC'$, ABC' and $A\,B'C$ are

straight line s which joins the point of intersection of AB and $A'B'$ to that of AB' and $A'B$

VI If from different points of a straight line s pairs of tangents a and a', b and b', c and c', be drawn to the conic, these form an involution For if A and A', B and B', C and C', are the points of contact of the tangents a and a', b and b', c and c', respectively, and S is the point of intersection of the chords AA' and BB', then in the involution determined by the pairs A, A' and B, B' the straight line joining any other pair of conjugate points will pass through S The point C and its conjugate lie therefore on a straight line passing through S, and the tangents at these points must meet on the straight line joining the points aa' and bb', *i.e.* on s, the conjugate of C is therefore C' This shows that A and A', B and B', C and C' form a range of points in involution, and that consequently a and a' b and b', c and c' form a series of tangents in involution.

VII If M and N are the double points of an involution AA', BB', CC', of points on a conic, it has been seen that AB, $A'B'$, MN are three concurrent straight lines (the same is the case with regard to AB', $A'B$, MN) In consequence then of theorem V, above, we conclude that

If AA' and BB' are two pairs of conjugate elements of an [involu]tion, and MN the double elements, then MN, AB, and A'[B'] [(]similarly MN, AB', and $A'B$) are three pairs of conjugate ele[ments] of another involution

VIII The straight line s cuts the conic (see below, Ar[t.] 254) when the point S lies outside the conic (Fig 138), that is

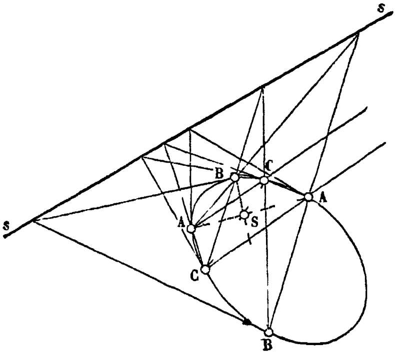


Fig 139

when the arcs AA' and BB' do not overlap one another, when these arcs overlap, the point S lies within the conic and the

arrive again at the property already proved in Art 128, viz. that

An involution has two double elements when any two pairs of conjugate elements are such that they do not overlap, and it has no double elements when they are such that they do overlap

In no case can an involution, properly so called, have only one double element For if s were a tangent to the conic, S would be its point of contact, and of every pair of conjugate points one would coincide with S (cf Art 125)

204. If $(MNAB\ \ )$ and $(MNA'B'\ \ )$ are two projective ranges of points on a conic, M and N will be the self-corresponding points, and the straight line MN will pass through the point of intersection of AB' and $A'B$ (Art 200) Now let B' be supposed to lie indefinitely near to A and similarly B to A', so that the straight lines AB' and $A'B$ become in the limit the tangents at A and A' respectively (Fig 140) Since now $MNAA'$ and $MNA'A$ are groups of corresponding points of two projective ranges, the two pencils $mnaa'$ and $mna'a$ formed by joining them to any point O on the conic will be projective, and therefore $mnaa'$ is a harmonic pencil (Art 83) We thus arrive again at the second theorem of Art 195 (right), viz

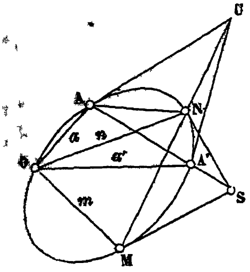


Fig 140

If four points M, N, A, A' on a conic are harmonic, the tangents at one pair of conjugate points, say A and A', intersect on the chord MN joining the other pair,

and its correlative (Art 195, left),

If four tangents to a conic are harmonic, the point of intersection of one pair of conjugates lies on the chord of contact of the other pair

From the former of these it follows that if through the point of intersection S of the tangents at M and N straight lines be drawn cutting the conic in A and A', B and B', C and C', respectively, any of these pairs of points will be harmonically conjugate with regard to M and N The tangents at A and A', B and B', C and C', will therefore intersect in

In other words

If from any point there be drawn to a conic two tangents a a secant, the two points of contact and the two points of intersection form a harmonic system

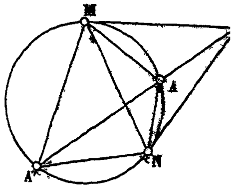


Fig 141

The points (AA'), (BB'), (CC'), form an involution of which M and N are the double points (Art 203, III, IV) We therefore arrive again at the property of an involution that if it has two double elements these are separated harmoi cally by any pair of conjugate elements (Art. 125)

205 Suppose now that the conic is a circle (Fig 141) From t similar triangles SAM, SMA',

$$AM\ \ MA' \quad SM\ \ SA',$$

and from the similar triangles SAN, SNA'

$$AN\ \ NA' \quad SN\ \ SA',$$

$$\frac{AM}{AN}=\frac{A'M}{A'N}, \text{ (since } SM=SN),$$

or

$$AM\ \ A'N=AN\ \ A'M$$

But by Ptolemy's theorem (Euc vi D),

$$AA'\ \ MN=AM\ \ A'N+AN\ \ A'M$$

If then M, N, A, A' are four harmonic points on a circle,

$$\tfrac{1}{2}AA'\ \ MN=AM\ \ A'N=AN\ \ A'M$$

206 The properties established in Art 200 and the followi Articles lead at once to the solution of the important problem

To construct the self-corresponding elements of two superposed p jective forms, and the double elements of an involution

I *Let two concentric projective pencils be given, which are det mined by three pairs of corresponding rays* (Fig 142), *it is required to construct their self-corresponding rays*

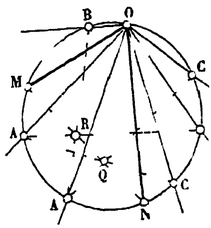


Fig 142

Through the common centre O describe any circle, cutting the three given pairs of rays in A and A', B and B', C and C' respectively Let AB', $A'B$ meet in R, and AC', $A'C$ in Q, if the straight line QR cut the circle in two points M and N, then OM, ON will be the required self-

II. *Let A and A′, B and B′, C and C′* (Fig 143) *be three pairs of corresponding points of two collinear ranges, it is required to construct the self-corresponding points.*

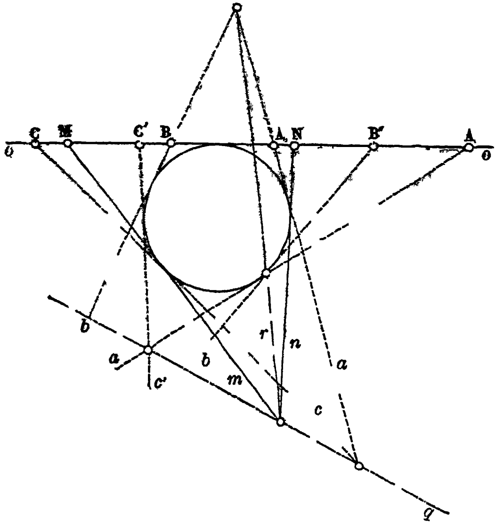


Fig 143

Describe any circle touching the common base o of the two ranges, and to this circle draw from the given points the tangents a and a', b and b', c and c' Let r be the straight line which joins the points ab', $a'b$, and q that which joins the points ac', $a'c$ If the point qr lies outside the circle and from it the tangents m and n be drawn to the circle, then the points om, on in which these meet the base will be the required self corresponding points of the two ranges

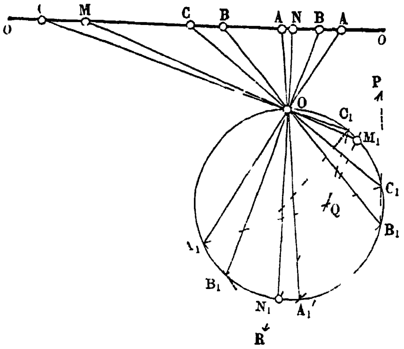


Fig 144

Otherwise (Fig 144)

O From O project the given points upon the circumference of circle, and let A_1 and A_1', B_1 and B_1', C_1 and C_1' be the projections of A and A', B and B', C and C' respectively. Join A_1B_1', A meeting in R, and A_1C_1', $A_1'C_1$ meeting in Q (or B_1C_1', $B_1'C_1$ meeting in P) If the straight line PQR cut the circle in two points M_1, N_1, and these be projected from the point O back upon the given base o, then their projections M, N will be the required self-corresponding points of the given ranges *

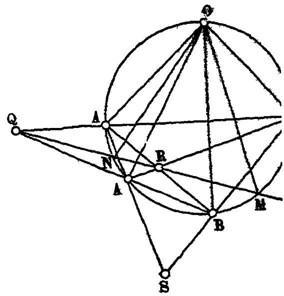


Fig 145

III *In* (I) *let the two pencils be in involution* (Fig 145), *and let it be required to find the double rays*

Two pairs of conjugate rays suffice now to determine the pencils Draw through the centre O any circle cutting the given rays in A and A', B and B' respectively Let AB', $A'B$ meet in R, and AB, $A'B'$ in Q, if the straight line QR cut the circle in two points M and N, then OM, ON will be the required double rays of the involution

IV *Let A and A', B and B' be two given pairs of conjugates of*

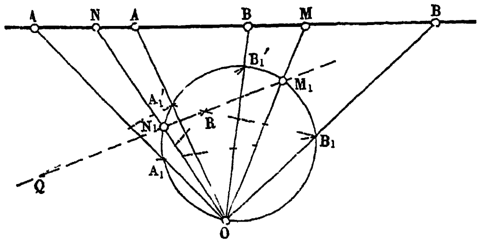


Fig 146

involution of points on a straight line, it is required to find double points (Fig 146)

Draw any circle in the plane and take on it any point O From O project the given points upon the circumference of the circle, and let A_1 and A_1', B_1 and B_1' be the projections of A and A', B and B' respectively Let A_1B_1', $A_1'B_1$ meet in R, and A_1B_1, $A_1'B_1'$ in Q If QR cuts the circle in M_1, N_1, and these points be projected from O back upon the given straight line, then their projections M, N will be the required double points

* Steiner *loc cit* pp 68 and 174 §§ 17 and 46 Collected Works, vo

Otherwise

Describe a circle touching the base AB (Fig 147), and draw to this circle from the points A and A', B and B', the tangents a and a',

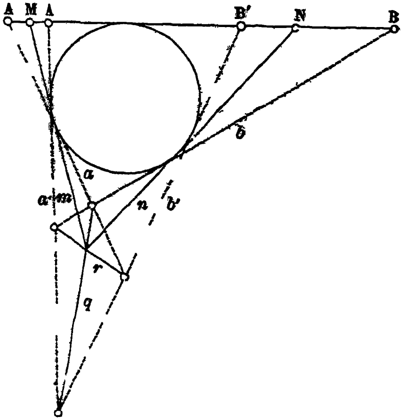


Fig 147

b and b', respectively Let r be the straight line which joins the points ab', $a'b$, and q that which joins the points ab, $a'b'$ If the point qr lies outside the circle, the tangents m and n from this point to the circle will cut the base line of the involution in the required double points

207 THEOREM *A pencil in involution is either such that every ray is at right angles to its conjugate, or else it contains one and only one pair of conjugate rays including a right angle*

Consider again Art 206, III, if the point of intersection S of the straight lines AA', BB', is the centre of the circle (Fig 148) then AA', BB', are all diameters, and therefore

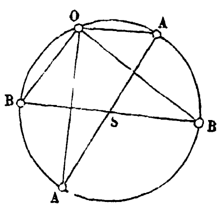


Fig 148

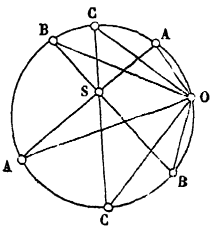


Fig 149

each ray OA, OB, will be at right angles to its conjugate OA', OB', In this case then the involution is formed by a

But if S is not the centre of the circle (Fig 149), dra[w] the diameter through it, if C and C' are the extremities [of] this diameter, the rays OC, OC' will include a right angl[e] But these will be the only pair of conjugate rays whic[h] possess this property, since through S only one diameter ca[n] be drawn

208 This proposition is only a particular case of th[e] following one

Two superposed involutions (or such as are contained in the sam[e] one-dimensional form) have always a pair of conjugate elements [in] common, except in the case where the involutions have doub[le] elements and the double elements of the one overlap those of the othe[r]

Take two involutions of rays having a common centre [O] and let a circle drawn through O cut the pairs of con[-]jugate rays of the first involution in the pairs of poin[ts] (AA', BB',) and those of the second in (GG', HH',) L[et] S be the point of intersection of AA', BB', and T that [of] GG', HH', If the straight line ST cut the circle in tw[o] points E and E', these will be a conjugate pair of each invol[u-]tion, since they are collinear with S and with T also Let [us] now examine in what cases ST will cut the circle

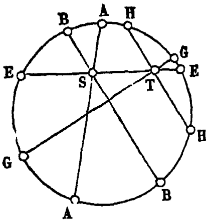


Fig 150

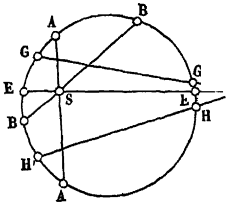


Fig 151

In the first place, it will certainly do so if one at least of th[e] points S, T lies within the circle (Art 203, VIII), *i e* if one a[t] least of the involutions has no double elements (Figs 150, 151)

Secondly, if both the points S, T lie outside the circle, *i e* [if] both the involutions have double elements, then the straigh[t] line ST may or may not cut the circle If OM, ON are th[e] double elements of the first involution, OU, OV those of th[e] second, the rays OE, OE' must be harmonically conjugate bot[h] with regard to OM, ON and with regard to OU, OV, but (Ar[t]

at the same time harmonically conjugate with regard to
ı of the two pairs OM, ON and OU, OV, it is necessary and
icient that these two pairs should not overlap If then
se pairs do not overlap, ST will cut the circle (Fig 152),

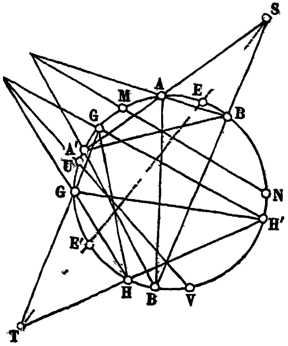


Fig 152

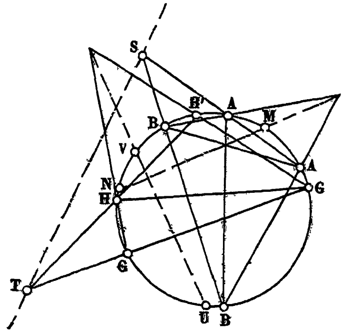


Fig 153

reas if they do overlap, ST will not cut the circle (Fig
) The two involutions have therefore a common pair of
ugate elements in all cases except this last viz when they
ı have double elements and these overlap

n Figs 150, 151 and 152, are shown cases of two involutions
ng a common pair of conjugate elements E and E', Fig 153
ne other hand illustrates the case where no such pair exists]

)9 The preceding problem, viz that of determining the common
of conjugate elements of two involutions superposed one upon
other, depends upon the following, *viz to determine* (in a range,
pencil, or on a conic) *a pair of elements which are harmonically
igate with regard to each of two given pairs* This problem has
dy been solved, for the case of a range, in Art 70, the following
other solution

ippose that we have to deal with a range of points lying on a
ght line Take any circle and a point O on it, and project the
ı points from O upon the circumference, let M, N and U, V be
projections (Fig 152) Let the tangents at M and N to the
meet in S, and the tangents at U and V in T If the pair MN
not overlap the pair UV, then ST will cut the circle in two
s E and E', which when projected back from O upon the given
ght line will give the points required

0 The double points of the involution determined by the pairs
' and B B' are the common pair of conjugate elements of two

and A', B', the other by the pairs A, B' and A', B (Art. 203, VII)

From this follows a *construction for the double points of an involution of collinear points which is determined by the pairs* A, A' *and* B, B' Take any point G outside the base of the involution and describe the circles GAB, $GA'B'$, they will meet in another point, say in H Similarly let K be the second point of intersection of the circles GAB', $GA'B$ Every circle passing through G and H meets the base in a pair of conjugate points of the involution AB, $A'B'$ (Art 127), so too every circle passing through G and K gives a pair of conjugate points of the involution AB', $A'B$ If then the circle GHK be described and it meet the base, the two points of intersection will be the double elements of the involution AA', BB'*

211. It follows from the foregoing that the determination of the self-corresponding points of two projective ranges ABC and $A'B'C'$ on a conic (and consequently of the self-corresponding points of any two superposed projective forms) reduces to the construction of the straight line s on which intersect the pairs of straight lines AB' and $A'B$, AC' and $A'C$, BC' and $B'C$, Similarly the determination of the double points of an involution AA', BB', depends on the construction of the straight line s on which intersect the pairs of straight lines AB and $A'B'$, AB' and $A'B$, or the pairs of tangents at A and A', B and B',

Conversely, if any straight line s (which does not touch the conic) is given, an involution of points on the conic is thereby determined, for it is only necessary to draw, from different points of s, pairs of tangents to the conic, and the points of contact will be pairs of conjugate points of an involution

But, on the other hand, in order that two projective ranges of points ABC and $A'B'C'$ may be determined, there must be given, in addition to the straight line s, a pair of conjugate points A and A' also, then the straight lines joining A and A' to any point on s will cut the conic in a pair of corresponding points B' and B

Two projective ranges of points determine an involution, for they determine the straight line s, which determines the involution If the two ranges have two self corresponding points, these will also be the double points of the involution

* CHASLES, *Geométrie supérieure*, Art 263

CHAPTER XIX

PROBLEMS OF THE SECOND DEGREE

212 PROBLEM *Given five points O, O', A, B, C on a conic, to determine the points of intersection of the curve with a given straight*

of

he

the

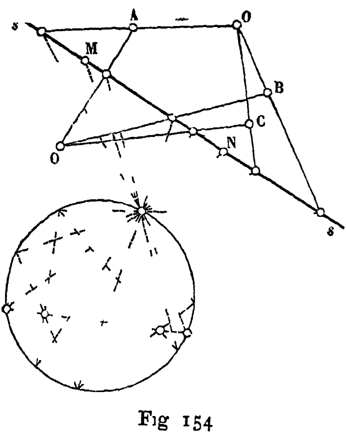


Fig 154

pencils O $(A, B, C, \;)$ and O' $(A, B, C, \;)$ will be projective, and will cut the transversal s in points forming two collinear projective ranges

A point M which corresponds

PROBLEM *Given five tangents o, o', a, b, c to a conic, to draw a pair of tangents to the curve from a given point S*

Consider the points where two of the tangents o, o' are met by the others a, b, c (Fig 155), the

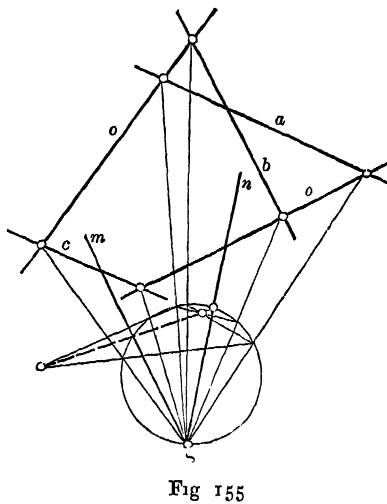


Fig 155

ranges o $(a, b, c, \;)$ and o' $(a, b, c, \;)$ will be projective and if projected from S as centre will give two concentric projective pencils

Any ray m which corresponds

also be a point on the conic, since a pair of corresponding rays of the two pencils must meet in M The points of intersection of the conic with the straight line s are therefore found as the self-corresponding points of the two collinear ranges which are determined on s by the three pairs of corresponding rays OA and $O'A$, OB and $O'B$, OC and $O'C$ There may be two such self-corresponding points, or only one, or none at all, consequently the straight line s may cut the conic in two points, or it may touch it, or it may not meet it at all The construction of the self-corresponding points themselves may be effected by either of the methods explained in Art 206, II

213 In a similar manner the problem may be solved if there be given four points O, O', A, B on a conic and the tangent o at one of them O, *or* three points O, O', A and the tangents o and o' at two of them O and O' In the first case the two pencils are determined by the three pairs of rays o and $O'O$, OA and $O'A$, OB and $O'B$, and in the second case by the three pairs o and $O'O$, OO' and o', OA and $O'A$

If however there be given five tangents, *or* four tangents and the point of contact of one of them, *or* three tangents and the points of contact of two of them, we may begin by first constructing such of the points of

also be a tangent to the conic, since a pair of corresponding points of the two ranges o and o' must lie on m. The tangents from S to the conic are therefore found as the self-corresponding rays of the two concentric pencils which are determined by the rays joining S to the three pairs of corresponding points oa and $o'a$, ob and $o'b$, oc and $o'c$. There may be two such self-corresponding rays, or only one, or none at all, consequently there can either be drawn from the point S two tangents to the conic, or S is a point on the conic, or else from S no tangent at all can be drawn. The construction of the self-corresponding rays themselves may be effected by the method explained in Art 206, I

In a similar manner the problem may be solved if there be given four tangents o, o', a, b to a conic and the point of contact O of one of them o, *or* three tangents o, o', a and the points of contact O and O' of two of them o and o' In the former case the three pairs of points which determine the two ranges are O and $o'o$, oa and $o'a$, ob and $o'b$, in the latter case they are O and $o'o$, oo' and O', oa and $o'a$

If however there be given five points on the conic, *or* four points and the tangent at one of them, *or* three points and the tangents at two of them, we may begin by first constructing such of the tangents at the points as are

already given (Arts 180, 171, 177), the problem will then reduce to one of the cases given above

171, 175), the problem will then reduce to one of the cases given above

214 In the construction given in Art 212 (left) suppose that the conic is a hyperbola and that the given straight line s is one of the asymptotes (Fig 156) The collinear projective ranges determined on s by the pencils $O(A, B, C, \quad)$ and $O'(A, B, C, \quad)$ will have in this case one self-corresponding point, and this (being the point of contact of the hyperbola and the asymptote) will lie at an infinite distance But in two collinear ranges whose self corresponding points coincide in a single one at infinity, the segment intercepted between any pair of corresponding points is of constant length (Art 103) We therefore conclude that

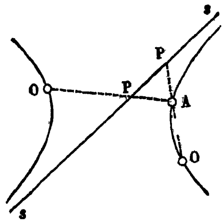


Fig 156

If from two fixed points O and O' on a hyperbola there be drawn two rays to cut one another on the curve, the segment PP' which these intercept on either of the asymptotes is of constant length *

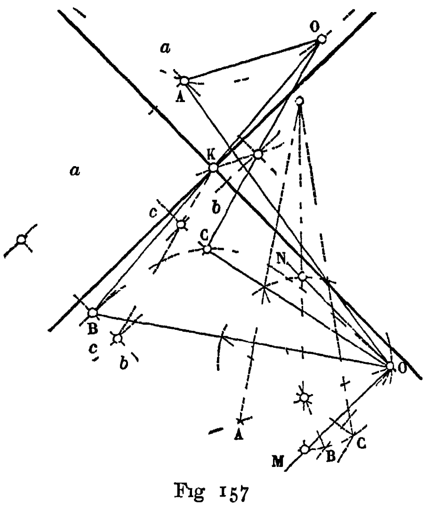


Fig 157

215 If in Art 212 (left) the straight line s be taken to lie at infinity, the problem becomes the following

Given five points O, O', A, B, C on a conic, to determine the points at infinity on it (Fig 157)

Consider again the projective pencils $O(A, B, C, \quad)$ $O'(A, B, C, \quad)$, which determine on the straight line at infinity s collinear ranges whose self-corresponding points are the requi points at infinity on the conic. Since each of these self-corresponc points must lie not only at the intersection of a pair of correspond rays of the two pencils but also on the line at infinity s, the co sponding rays which meet in such a point must be parallel to another, the problem therefore reduces to the determination of pairs of corresponding rays of the two pencils which are paralle one another

In order then to solve the problem we draw through O the paral OA', OB', OC' to $O'A$, $O'B$, $O'C$ respectively, and then const (Art 206, I) the self-corresponding rays of the two concen pencils which are determined by the three corresponding pairs and OA', OB and OB', OC and OC' If there are two self-co sponding rays OM and ON, the conic determined by the five gi points is a hyperbola whose points at infinity lie in the direct OM, ON, *i e* whose asymptotes are parallel to OM and respectively

If there is only one self-corresponding ray OM, the conic de mined by the five given points is a parabola whose point at infi lies in the direction OM

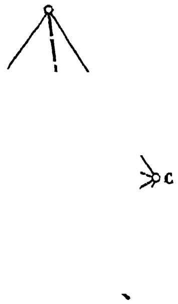
If there is no self corresponding ray, the conic determined by five given points is an ellipse, since it does not cut the straight at infinity

If in the first case (Fig 157) it is desired to construct the asy totes themselves of the hyperbola, we consider this latter as determi by the two points at infinity and three other points, say A, B, C, in other words, we regard the hyperbola as generated by the projective pencils, one of which consists of rays all parallel to (and the other of rays all parallel to ON, and which are such that pair of corresponding rays meet in A, a second pair in B, an third pair in C The rays which correspond in the two pen respectively to the straight line at infinity (the line joining centres of the pencils) will be the asymptotes required

Let then a, b, c (Fig 157) be the rays parallel to OM which] through A, B, C respectively, and let a', b', c' be the rays par to ON which pass through the same points respectively Join points ab' and $a'b$ and the points bc' and $b'c$, and let K be the p of intersection of the joining lines, the straight lines drawn thro K parallel to OM and ON will be the required asymptotes

216 PROBLEM *Given five points A, B, C, D, E on a conic draw the tangents from a given point S to the conic*

(left), by making use of the properties of the involution (Art 203) obtained by cutting the conic by transversals drawn through S



Join SA, SB (Fig 158), these straight lines will cut the conic again in two new points A' and B', which can be determined (making use of the ruler only, and without drawing the curve) by means of Pascal's theorem (Art 161, right) (In the figure the points A' and B' have been constructed by means of the hexagons $ADCBEA'$ and $BECADB'$ respectively) Now let the point of intersection of AB and $A'B'$ be joined to that of AB' ne s will pass through the points of contact of Art 203) The problem therefore reduces to e points of intersection of the conic and the 12, left)

a, *To find the points of intersection of a given straight line s and a conic which is determined by five given tangents*, may similarly be made to depend on that of Art 212 (right), by making a construction (Fig 159) analogous to the foregoing one

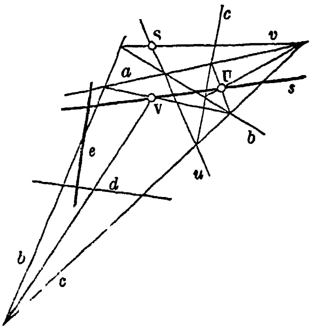


Fig 159

And the problem, *To draw through a given point a straight line which shall divide a given triangle into two parts having to one another a given ratio*, may be solved by reducing it to the following construction To draw from the given point a tangent to a hyperbola of which the asymptotes and a tangent are known

These are left as exercises to the student

218 PROBLEM *To construct a conic which shall pass through four given points Q, R, S, T, and shall touch a given straight line s which does not pass through any of the given points*

To construct a conic which shall touch four given straight lines q, r, s, t, and shall pass through a given point S which does not lie on any of the given lines

Solution Let A, A', B, B' ... Let a, a', b, b' be the ra...

be the points where the sides QT, RS, QR, ST respectively of the quadrangle $QRST$ cut the straight line s (Fig 160)

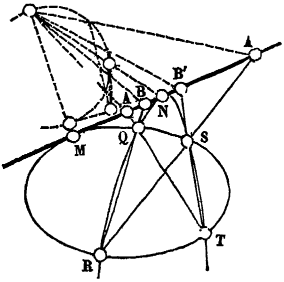


Fig 160

Construct the double points (if such exist) of the involution determined by the pairs of points A and A', B and B'

If there are two double points M and N, each of them will be (Art 185, left) the point of contact with s of some conic circumscribed about the quadrangle $QRST$ Each of the conics $QRSTM$, $QRSTN$ therefore gives a solution of the problem, and these conics can be constructed by points by help of Pascal's theorem (Art 161, right)

If however there are no double points, there is no conic which satisfies the conditions of the problem

joining the point S to the vert qt, rs, qr, st respectively of quadrilateral $qrst$ (Fig 1 Construct the double rays

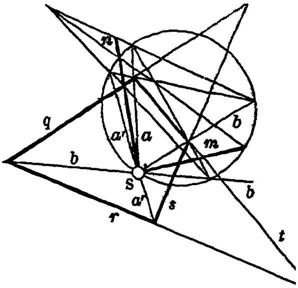


Fig 161

such exist) of the involution termined by the pairs of 1 a and a', b and b'

If there are two double 1 m and n, each of them will (Art 185, right) a tangent S to some conic inscribed the quadrilateral $qrst$ Each the conics $qrstm$, $qrstn$ there gives a solution of the probl and these conics can be c structed by tangents by help Brianchon's theorem (Art 1 left)

If however there are no dou rays, there is no conic wh satisfies the conditions of problem

219 If in the foregoing Art (left) the straight line s be take lie at infinity, the problem becomes the following

To construct a parabola which shall pass through four given poi Q, R, S, T

To solve it, take any point O (Fig 162), and through it d the rays a, a', b, b' parallel respectively to the straight l QT, RS, QR, ST, and construct the double rays (if such ex

Each of these double rays will determine the direction in which lies the point at infinity on a parabola passing through the four given points, the problem therefore reduces to the last problem of Art 165 If however the involution has no double rays, no parabola can be found which satisfies the conditions of the problem

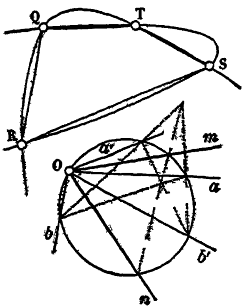


Fig 162

Through four given points therefore can be drawn either two parabolas or none, in the first case the other comes which pass through the given points are ellipses and hyperbolas, in the second case they are all hyperbolas The first case occurs when each of the four points lies outside the triangle formed by the other three (*i. e.* when the quadrangle formed by the four points is *non-reentrant*), the second case when one of the four points lies within the triangle formed by the other three (*i e.* when the quadrangle formed by the four points is *reentrant*)

220 If in Art. 218 (right) one of the straight lines q, r, s, t lies at infinity, the problem becomes the following

To construct a parabola which shall touch three given straight lines and shall pass through a given point

221 Problem *To construct a conic which shall pass through three given points P, P', P'' and shall touch two given straight lines q and s, neither of which passes through any of the given points*

Solution This depends on the theorem of Art 191 (left) Join PP', and consider it as a transversal which cuts the conic in P and P', and the pair of tangents q and s in the two points B and B' (Fig 163) If A and A_1 are the double points of the involution determined by the two pairs of points P and P', B and B', the chord of contact of the conic and the tangents q and s must pass through one of these points

To construct a conic which shall touch three given straight lines p, p', p'' and shall pass through two given points Q and S, neither of which lies on any of the given straight lines

The solution depends on the theorem of Art 191 (right) Consider pp' as a point from which the tangents p and p' have been drawn to the conic, and the rays b and b' to the two points Q and S (Fig 164) If a and a_1 are the double rays of the involution determined by the two pairs of rays p and p', b and b', the point of intersection of the tangents at Q and S to the conic must lie on one of these rays by the theorem

Repeat the same reasoning for the case of the transversal PP'', which cuts q and s in D and D'',

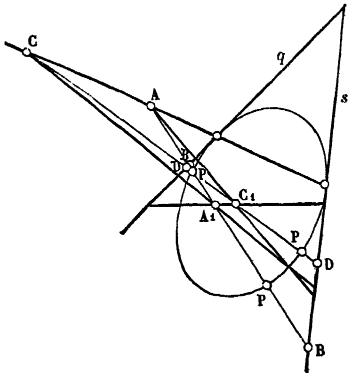


Fig 163

if C and C_1 are the double points of the involution determined by the two pairs of points P and P'', D and D'', the chord of contact must similarly pass through C or C_1 The problem admits therefore of four solutions, *viz* when the two involutions (PP', BB') and (PP'', DD'') both have double points, there are four conics which satisfy the given conditions If the double points are A, A_1 and C, C_1 respectively, the chords of contact of the four conics and the tangents q and s are AC, A_1C, AC_1, and A_1C_1 Of each of these conics five points are known, *viz* P, P', P'', and the two points of intersection of AC (or of A_1C, or AC_1, or A_1C_1, as the case may be) with q and s, they can accordingly be constructed by points by means of Pascal's theorem

reasoning for the case of the poi pp'', from which are drawn t rays d and d'' to the poi

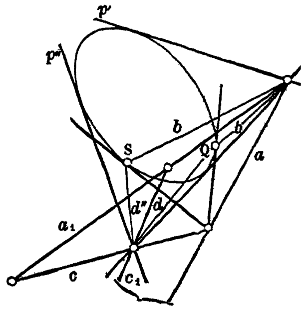


Fig 164.

Q and S, if c and c_1 are t double rays of the involution d termined by the two pairs o p and p'', d and d'', the poi intersection of the tangents m similarly lie on c or c_1 T problem admits therefore of fo solutions, *viz* when the two volutions (pp', bb') and $(pp'', dd$ both have double rays, there a four conics which satisfy the giv conditions If the double rays a a, a_1 and c, c_1 respectively, t points of intersection of t tangents at Q and S to the fo conics are ac, a_1c, ac_1, and a_1 Of each of these conics fi tangents are known, viz p, p', p and the two straight lines whi join ac (or a_1c, or ac_1, or a_1c_1, the case may be) to Q and they can accordingly be co structed by tangents by means Brianchon's theorem (Art 16

222. Problem *To construct a polygon whose vertices shall lie on given straight lines (each on each), and whose sides shall pass through given points (each through each*)*

Solution For the sake of simplicity suppose that it is required to construct a quadrilateral, whose vertices 1, 2, 3, 4 shall lie respectively on four given straight lines s_1, s_2, s_3, s_4, and whose sides 12, 23, 34, 41 shall pass respectively through four given points S_{12}, S_{23}, S_{34}, S_{41} (Fig 165) The method and reasoning will be the

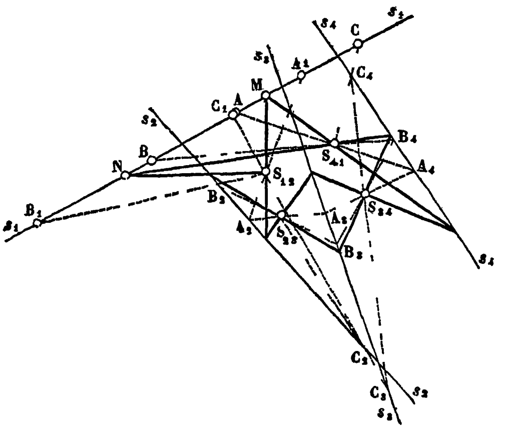


Fig 165

same as for a polygon of any number of sides Take any points A_1, B_1, C_1, on s_1 and project them from S_{12} as centre upon s_2; and let A_2, B_2, C_2, be their projections Project A_2, B_2, C_2, from S_{23} as centre upon s_3, and let A_3, B_3, C_3, be their projections Project A_3, B_3, C_3, from S_{34} as centre upon s_4, and let A_4, B_4, C_4, be their projections Finally project A_4, B_4, C_4, from S_{41} as centre upon s_1, and let A, B, C, be their projections

The points S_{12}, S_{23}, S_{34}, S_{41} are the centres of four projectively related pencils, for the first and second are in perspective (since their pairs of corresponding rays A_1A_2, B_1B_2, and A_2A_3, B_2B_3, intersect on s_2), the second and third are in perspective (pairs of corresponding rays intersect on s_3), and similarly the third and fourth are in perspective (pairs of corresponding rays intersect on s_4) Consequently (Art 150) pairs of corresponding rays of the first and fourth pencils (such as A_1A_2 and A_4A) will intersect on a conic, or in other words the locus of the first vertex of the variable quadrilateral whose second, third, and fourth vertices (A_2, A_3, A_4) slide respectively on three given straight lines (s_2, s_3, s_4) and whose sides (A_1A_2, A_2A_3, A_3A_4, A_4A) pass respectively through four given points

is a conic* This conic passes through the points S_{12}, S_{41}, t centres of the pencils which generate it, in order therefore to det mine it, three other points on it must be known, the intersectio of the three pairs of corresponding rays A_1A_2 and A_4A, B_1B_2 and B_4 C_1C_2 and C_4C will suffice It is then only necessary further to c struct (Art 212) the points of intersection M and N of the straig line s_1 with the conic determined by these five points, either M or can then be taken as the first vertex of the required quadrilateral.

This construction may be looked at from another point of view T broken lines $A_1A_2A_3A_4A$, $B_1B_2B_3B_4B$, and $C_1C_2C_3C_4C$ may be regar as the results of so many attempts made to construct the required qu rilateral, these attempts however give polygons which are not clos for A does not in general coincide with A_1, nor B with B_1, C with C_1 These attempts and all other conceivable ones which mi similarly be made, but which it is not necessary to perform, give the straight line s_1 two ranges $A_1B_1C_1$ and ABC, one be traced out by the first vertex and the other by the last vertex of open polygon These ranges are projective with one another, s the second has been derived from the first by means of project from S_{12}, S_{23}, S_{34}, S_{41} as centres, and sections by the transver s_2, s_3, s_4, s_1 Each of the self-corresponding points therefore of two ranges will give a solution of the problem, for, if the first ver of the polygon be taken there, the last vertex will also fall on same point, and the polygon will be closed

In the following examples also the method remains the whatever be the number of sides of the polygon which it is required to construct

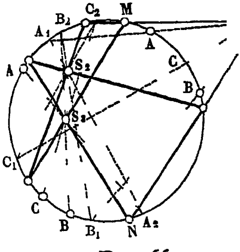


Fig 166

223 Problem *To inscribe in a given† conic a polygon whose sides pass respectively through given points*

Solution Suppose that it is required to inscribe in the conic a triangle whose sides pass respectively through three given points S_1, S_2, S_3 (Fig 166) Let us make three trials Take then any three po A, B, C on the conic, join them to S_1 and let the joining li cut the conic again in A_1, B_1, C_1, join these points to S_2 and

* This theorem, *viz* that 'if a simple polygon move in such a way tha sides pass respectively through given points and all its vertices except one s respectively along given straight lines, then the remaining vertex will descri conic, is due to Maclaurin (Phil Trans, London, 1735) Cf Chasles, *Ap*

the joining lines cut the conic again in A_2, B_2, C_2, finally join these points to S_3 and let the joining lines cut the conic again in A', B', C' Since the point finally arrived at, A' or B' or C', does not in general coincide with the corresponding starting-point A or B or C, we shall have, instead of an inscribed triangle as required by the problem, three polygons AA_1A_2A', BB_1B_2B', CC_1C_2C' which are not closed. But since, by a series of projections from S_1, S_2, S_3 in succession as centres, we have passed from the range A, B, C, to the range A_1, B_1, C_1, from this last to A_2, B_2, C_2, ..., and from this to A', B', C', , it follows that the range of points A, B, C, , with which we started is projective with the range of points A', B', C', , with which we ended (Arts 200, 201, 203). The problem would be solved if one of the points in the latter range coincided with its correspondent in the former If then the two projective ranges ABC and $A'B'C'$ have self-corresponding points, each of these may be taken as the first vertex of a triangle which satisfies the given conditions. We have therefore only to determine (Art. 200, II) the straight line on which intersect the three pairs of opposite sides of the inscribed hexagon $AB'CA'BC'$, and to construct (Art. 212) the points of intersection M and N of this straight line with the conic, each of them will give a solution of the problem*

224. By a similar method may be solved the correlative problem

To circumscribe about a given conic (i e one which is either completely drawn or determined by five tangents) a polygon whose vertices lie respectively on given straight lines

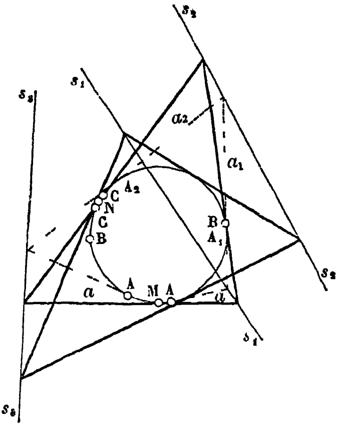


Fig 167

Suppose that it is required to circumscribe about the conic a triangle whose vertices lie respectively on the straight lines s_1, s_2, s_3 (Fig 167) Take any point A on the conic and draw the tangent a at it, from the point where this tangent cuts s_1 draw another tangent a_1 (let its point of contact be A_1), from the point where a_1 cuts s_2 draw a third tangent a_2 (let its point of contact be A_2), finally, from the point where a_2 cuts s_3 draw the tangent a', and let its point of contact be A' The problem would be solved if the point A'

* PONCELET, *loc cit*, p 352

coincided with A, i. e. if the tangents a' and a coincided with o[ne] another Suppose that other similar trials have been made, takin[g] other arbitrary points B, C, on the conic to begin with, then [we] shall arrive in succession at the ranges of points A, B, C, .. A_1, B_1, C_1, , A_2, B_2, C_2, , and A', B', C', , which are [all] projectively related to one another For the first range is projectiv[e] with the second (Art. 203), since the tangents at A and A_1, B and B[$_1$] C and C_1, always intersect on s_1, and for similar reasons t[he] second and third, and the third and fourth, are projective with e[ach] another, consequently (Art. 201) the same is true of the fourth a[nd] the first Since the problem would be solved if A' coincided with [A] or B' with B, , each of the self-corresponding points of the pr[o]jective ranges ABC and $A'B'C'$ may be taken as the point [of] contact of the first side of a triangle which satisfies the given co[n]ditions We have therefore only to make three trials (Art. 20[6]) i e. to take any three points A, B, C on the conic and to deri[ve] from them the corresponding points A', B', C', and then to co[n]struct the points of intersection of the conic with the straight li[ne] which joins the points of intersection of the three pairs of oppos[ite] sides (the Pascal line) of the inscribed hexagon $AB'CA'BC'$*

225 The particular case of the problem of Art. 223 in which t[he] given points S_1, S_2, lie all upon one straight line s must be co[n]sidered separately If the number of sides of the required polyg[on] is even, the theorem of Art 187 may be applied, in this case t[he] problem has either no solution at all, or it has an infinite number [of] solutions Suppose it required, for example, to inscri[be] an octagon of which the first seven sides pass respective[ly through] the points S_1, S_2, S_7, then by the theorem just quoted the last si[de] will pass through a fixed point S on s this point S is not arbitra[ry] but its position is determined by those of the points S_1, S_2, [...] If then the last of the given points S_8 coincides with S, there are [an] infinite number of octagons which satisfy the given conditions B[ut] if S_8 does not coincide with S, there is no solution

If the number of sides of the required polygon is odd, the probl[em] becomes determinate Suppose it is required to inscribe in the coni[c a] heptagon (Fig 124) whose sides pass respectively through the giv[en] collinear points S_1, S_2, S_3, S_7 By the theorem of Art 187 the[re] exist an infinite number of octagons whose first seven sides pass throu[gh the] seven given collinear points and whose eighth side passes throu[gh] a fixed point S collinear with the others If among these octag[ons] there is one such that its eighth side touches the conic, the probl[em] will be solved, for this octagon, having two of its vertices indefinit[ely]

* PONCELET, *loc cit*, p 3.4

near to one another, will reduce to an inscribed heptagon, whose sides pass respectively through seven given points If then tangents can be drawn from the point S to the conic, the point of contact of each of them will give a solution (Art. 187) According therefore to the position of the point S with reference to the conic, there will be two solutions, or only one, or none

In Fig 126 is shown the case of this problem where the polygon to be inscribed is a triangle *

The solution of the correlative problem, *to circumscribe about a given conic a polygon whose vertices lie respectively on given rays of a pencil*, is left as an exercise to the student. This problem also is either indeterminate or impossible if the polygon is one of an even number of sides, it is determinate and of the second degree if the polygon is one having an odd number of sides (Figs. 125, 127)

226 LEMMA. *If two conics cut one another in the points A, B, C, C', and if from A and B respectively two straight lines AFF', BGG' be drawn cutting the first conic in F and G, and the second in F' and G', then the chords FG, $F'G'$ will intersect in a point H lying on the chord CC'* (Fig 168)

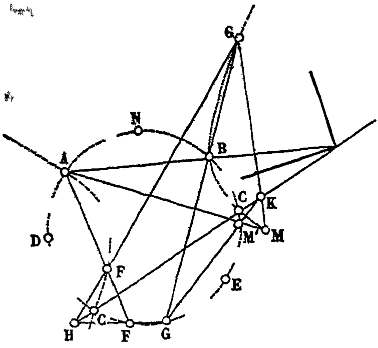


Fig 168

The transversal CC' cuts the first conic and the opposite sides of the inscribed quadrangle $ABGF$ in six points of an involution (Art 183, left), and the same is true with regard to the second conic and the inscribed quadrangle $ABG'F'$ But the two involutions must coincide (Art 127), since they have two pairs of conjugate points in common, *viz* the points C, C' in which the transversal cuts both the conics, and the points in which it cuts the pair of opposite sides AFF', BGG', which belong to both quadrangles The involutions will therefore have every pair of conjugate points in common, and therefore the transversal CC' will meet FG and $F'G'$ in the same point H, the conjugate of the point in which it meets AB†

227 The preceding lemma, which is merely a corollary of Desargues theorem, leads at once to the solution of the two following problems, one of which is of the first, and the other of the second degree

* PAPPUS, *loc cit*, book vii prop 117

† This may also be proved very simply by applying Pascal's theorem to each of the hexagons $AFGBCC'$, $AF'G'BCC'$ in turn

I. PROBLEM *Given three of the points of intersection A, B, C two conics, and in addition two other points D, E of the first, and t other points F, G of the second, to determine the fourth point of int section of the two conics* (Fig 168)

Take two of the given points of intersection A and B, and jo AF, BG These straight lines will cut the first conic again in poin F', G' respectively which can be determined by the method of A 161 (right) Join FG, $F'G'$, and let them meet in H By the fo going lemma H will lie on the chord joining the other two points intersection of the conics. This chord will therefore be HC, a it remains only to determine the point C' where HC cuts either the conics; C' will be the required fourth point of intersection of t conics.

II PROBLEM. *Given two of the points of intersection, A, B, of t conics, and in addition the three points D, E, N of the first and t three points F, G, M of the second, to determine the other two poi of intersection of the conics* (Fig 168)

Join AF and BG, and let them meet the first conic again in F', respectively; join FG, $F'G'$, and let them meet in H The point will lie on the chord joining the two required points Again, jo AM, and let it meet the first conic again in M'; join GM, $G'M'$, a let these meet in K; then the point K also will lie on the sa chord The required points therefore lie on HK, and the probl reduces to the determination (Art 212) of the points of intersecti C, C' of the conics with HK *

228 The solution just given of problem II holds good equally wh the points A and B lie indefinitely near to one another, *i e* two conics touch a given straight line at the same given point

In this case two conics are given which touch one another at point A, and the straight line HK is constructed which joins th remaining points of intersection C and C' If HK passes throu A, one of the points C or C' must coincide with A, since a co cannot cut a straight line in three points When this is the ca three of the four points of intersection of the conics lie indefinit near to one another, and may be said to coincide in the point and the conics are said to *osculate* at the point A The construct gives a point H of the chord which joins A to the fourth point intersection C of the conics It may happen that this chord co cides with the tangent at A; in this case A represents four coincid points of intersection of the two conics (or rather, four such poi lying indefinitely near to one another)

* GASKIN, *The geometrical construction of a conic section, &c* (Cambridge 18 pp 26, 40

229 Let now the lemma of Art. 226 be applied to the case of a conic and a circle touching it at a point A. At A draw the normal to the conic (the perpendicular to the tangent at A), and let it cut the conic again in F and the circle again in F'. On AF as diameter describe a circle; this circle, which touches the conic at A and cuts it at F, will cut it again at another point G such that AGF is a right angle. Join AG and let G' be the point where it cuts the first circle. Join FG, $F'G'$; by the lemma they will intersect on the chord HK; but they are parallel to one another, since $AG'F'$ also is a right angle. Thus *for any circle whatever which touches the conic at A, the chord of intersection HK with the conic has a constant direction, viz. that parallel to FG.*

If HK passes through A, the conic and the circle osculate at this point. If then a parallel through A to FG cut the conic again in C, the circle which touches the conic at A and cuts it at C will be the osculating circle (circle of curvature) at A.*

[In the particular case where A is a vertex (Art. 297) of the conic, F will be the other vertex, FG the tangent at F, AC the tangent at A, and C will coincide with A. It is seen then that the osculating circle at a vertex of a conic has not only three but *four* indefinitely near points in common with the conic.]

Conversely, the conic can be constructed which passes through three given points A, P, Q and has a given circle for its osculating circle at one of these points A.

For join AP, AQ, and let them cut the given circle in P', Q' respectively, and join PQ, $P'Q'$, meeting in U. If AU be joined and cut the circle again in C, the required conic will pass through C. It is therefore determined by the four points A, P, Q, C and the tangent at A (which is the same as the tangent to the circle there).

230 The proposition correlative to the lemma of Art. 226 may be enunciated as follows:

If a and b are a pair of common tangents to two conics, and if from two points taken on a and b respectively the tangents f, g be drawn to the first conic and the tangents f', g' to the second, then the points fg and $f'g'$ will be collinear with the point of intersection of the second pair of common tangents to the conics.

This proposition enables us to solve the problems which are correlative to I and II of Art. 227, viz. given three (or two) of the common tangents to two conics, and in addition two (or three) tangents to the first and two (or three) tangents to the second, to determine the remaining common tangent (or the two remaining common tangents) to the conics.

231 PROBLEM. *Given eleven points $A, B, C, D, E, A_1, B_1, C_1, D_1, E_1, P$*

* PONCELET, *loc. cit.*, Arts. 334–337.

to construct by points the conic which passes through P and through t four points of intersection of the two conics which are determined by t points A, B, C, D, E and A_1, B_1, C_1, D_1, E_1 respectively The coni are supposed not to be traced, nor are their points of intersection given

Solution. Draw through P any transversal, and construct (Ar 212, left) the points M and M' in which it cuts the conic $ABCD$ and the points N and N' in which it cuts the conic $A_1B_1C_1D_1E$ Since these two conics and the required one all pass through the sam four points, Desargues' theorem may be applied to them If therefo (Art 134, left) the point P' be constructed, conjugate to P in t involution determined by the pairs of points M and M', N and N this point P' will lie on the required conic. By causing the tran versal to turn about the point P, other points on the required con may be obtained

232 PROBLEM. *Given ten points A, B, C, D, E, A_1, B_1, C_1, D_1, and a straight line s; to construct a conic which shall touch s an shall pass through the four points of intersection of the two coni which are determined by the points A, B, C, D, E and A_1, B_1, C_1, D_1, respectively The conics are supposed not to be traced, nor are the points of intersection given*

Solution Construct (Art 212) the points of intersection M and M' of s with the conic $ABCDE$, and the points of intersection N a N' of s with the conic $A_1B_1C_1D_1E_1$, and then (Art 134) the dou points of the involution determined by the two pairs of M and M', N and N' If P is one of these double points, it the point of contact (Art 185) of s with a conic drawn through four points of intersection of the conics $ABCDE$ and $A_1B_1C_1L$ to touch s The problem thus reduces to that of the precedi Article

233 The correlative constructions give the solutions of the corr lative problems *viz* to construct a conic which passes through a giv point (or which touches a given straight line), and which is inscrib in the quadrilateral formed by the four common tangents to t conics, the conics being supposed each to be determined by five giv tangents, but not to be completely traced, and their four comm tangents being supposed not to be given

234 PROBLEM *Through a given point S to draw a straight li which shall be cut by four given straight lines a, b c, d in four poi having a given anharmonic ratio*

Solution It has been seen (Art 151) that the straight lin which are cut by four given straight lines in four points having given anharmonic ratio are all tangents to one and the same con

* PONCELET *loc cit*, Art 389

Over

touching the given straight lines, and that if A, B, C are the points where d cuts a, b, c respectively, and D is the point of contact of d, the anharmonic ratio $(ABCD)$ is equal to that of the four points in which the straight lines a, b, c, d are cut by any other tangent to the conic. Accordingly, if on the straight line d that point D be constructed (Art. 65) which gives with the points

$$ad\,(\equiv A)\,,\ bd\,(\equiv B)\,,\ cd\,(\equiv C)$$

an anharmonic ratio $(ABCD)$ equal to the given one, and if then the straight lines be constructed (Art 213, right) which pass through S and touch the conic determined by the four tangents a, b, c, d and the point of contact D of d, each of these straight lines will give a solution of the proposed problem.

If one of the straight lines a, b, c, d lie at infinity, the problem becomes the following

Given three straight lines a, b, c and a point S, to draw through S a straight line such that the segment intercepted on it between a and b may be to that intercepted on it between a and c in a given ratio

To solve this, construct on the straight line a that point A which is so related to the points $ab\,(\equiv B)$ and $ac\,(\equiv C)$ that the ratio AB AC has the given value, and draw from S the tangents to the parabola which is determined by the tangents a, b, c and the point of contact A of a

The correlative construction gives the solution of the following

ι On a given straight line s to find a point such that the ιning it to four given points A, B, C, D form a pencil having ι anharmonic ratio

235 Problem *Given two projective ranges of points lying on the straight lines u, u' respectively, to find two corresponding segments MP, $M'P'$ such that the angles MOP, $M'O'P'$ which they subtend at two fixed points O, O' respectively may be given in sign and magnitude*

Solution Take on u' two points A' and D' such that the angle $A'O'D'$ may be equal to the second of the given angles, let A and D be the points on u which correspond respectively to A' and D', and let A_1 be a point on u such that the angle A_1OD is equal to the first of the given angles The problem would evidently be solved if OA_1 coincided with OA, since in this case the angles AOD and $A'O'D'$ would be equal to the given angles respectively If the rays $O'A'$, OA, $O'D'$, OD, OA_1 be made to vary simultaneously, they will trace out pencils which are projectively related For those traced out by $O'A'$ and $O'D'$ respectively are projective, and similarly those traced out by OA_1 and OD respectively, since the angles $A'O'D'$ and A_1OD are constant (Art 108), and the pencils traced out by

OA and $O'A'$ respectively, and by OD, $O'D'$ respectively, are pro-jective since the given ranges on u and u' are so Consequently the pencils generated by OA and OA_1 respectively are projective, and their self-corresponding rays give the solutions of the problem. If three trials be made of a similar kind to the foregoing one, three pairs of corresponding rays OA and OA_1, OB and OB_1, OC and OC_1 will be obtained, let the self-corresponding rays of the concentric projective pencils determined by these three pairs be constructed (Art 206, I). If one of these self-corresponding rays meets u in M and if the point P be taken on u such that the angle MOP is equal to the first of the given ones, and if then on u' the points M', P' be found which correspond to M, P respectively, the angle $M'O'P'$ will be equal to the second of the given angles, and the problem will be solved.

236 Problem. *Given two projective ranges of points A, B, C, and A', B', C', lying on the straight lines u and u' respectively, to find two corresponding segments which shall be equal, in sign and magnitude, to two given segments.*

Solution Take on u' a segment $A'D'$ equal to the second of the given ones, and let AD be the segment on u which corresponds to $A'D'$ Take on u the point A_1 such that A_1D is equal to the first of the given segments, then the problem would be solved if A_1 coincide with A If the points A, A', D', D, A_1 be made to vary simulta-neously, the ranges traced out by A and A' respectively will be projec-tive with one another, as also those traced out by D and D' re (by reason of the projective relation existing between ABC $A'B'C'$), and the ranges traced out by A and D respectivel similarly those traced out by A' and D' respectively, will be proj with one another, since they are generated by segments of constant length sliding along straight lines (Art 103) Consequently also the ranges traced out by A and A_1 are projectively related, and their self-corresponding points give the solutions of the problem It is there-fore only necessary to obtain three pairs of corresponding poin A and A', B and B', C and C', by making three trials, and then construct the self-corresponding points of the ranges determined b these three pairs (Art 206, II)

237 The student cannot have failed to remark that the metho employed in the solution of the preceding problems has been in a cases substantially the same This method is general, uniform, an direct, and it may be applied in a more or less simple manner to a problems of the second degree, *i e* to all questions which when treate algebraically would depend on a quadratic equation It consists i making three trials, which give three pairs of corresponding elemen of two superposed projective forms, the self-corresponding elemen of these systems give the solutions of the problem This method

precisely analogous to that known in Arithmetic as the 'rule of false position,' and it has on that account been termed a *geometric method of false position* *

238 Problems of the second degree (and those which are reducible to such) are solved, like all those occurring in elementary Geometry, by means of the ruler and compasses only, that is to say by means of the intersections of straight lines and circles † But again, the solution of any such problem can be made to depend on the determination of the self-corresponding elements of two superposed projective forms, which determination depends (Art. 206) on the construction of the self-corresponding points of two projective ranges lying on a circle whose position and size is entirely arbitrary It follows that a single circle, described once for all, will enable us to solve all problems of the second degree which can be proposed with reference to any given elements lying in one plane (the plane in which the circle is drawn) ‡ This circle once described, any such problem will reduce to that of constructing three pairs of points of the two projective systems whose self-corresponding elements give the solution of the problem This done, we proceed to transfer to the circumference of the circle, by means of projections and sections, these three pairs of points. This will give three pairs of points on the circle, taking these as the pairs of opposite vertices of an inscribed hexagon, we have only further to draw the straight line which joins the points of intersection of the three pairs of opposite sides (the Pascal line) of this hexagon

It is hardly necessary to remark that instead of the solution of such a problem being made to depend on the common elements of two superposed projective forms, it may always be reduced to the determination of the double elements of an involution (Art 211)

Fig 169

The following Articles (239 to 249) contain examples of problems solved by means of the method just explained

239 Problem *Given* (Fig 169) *two projective ranges of points lying on the straight lines u and u′ respectively, and two other projective ranges of points lying on the straight lines v*

* Chasles, *Geom sup*, p 212

† A problem is said to be *of the first degree* when it can be solved with help of the ruler only, *i e* by the intersections of straight lines See Lambert, *loc cit* p 161 Brianchon, *loc cit*, p 6 Poncelet *loc cit* p 76

‡ Poncelet *loc cit*, p 187 Steiner *Die geometrischen Constructionen aus gefuhrt mittelst der geraden Linie und eines festen Kreises* (Berlin, 1833), p 67, Collected Works, vol i pp 461–522, Staudt *Geometrie der Lage* (Nurnberg, 1847), § 23

and v' respectively, it is required to draw through a given point O tw straight lines s and s', which shall cut u and u' in a pair of corr sponding points and also v and v' in a pair of corresponding points.

Through O draw any straight line cutting u', v' in A', P' respectively let A be the point on u which corresponds to A', and let P be th point on v which corresponds to P' The problem would be solved the straight lines OA and OP coincided with one another If the straight lines be made to change their positions simultaneously, the will trace out two concentric projective pencils (determined by thr trials of a similar kind to the one just made), and the self-corr sponding rays of these pencils will give the solutions of the problem

240 In the preceding problem the straight lines u and u' mig be taken to coincide, and similarly v and v' If all four straig lines coincided with one another, the problem would become t following

Given two projective ranges u, u' and two other projective rang v, v' all lying on one straight line, to find a pair of points which sh correspond to one another when regarded as points of the ranges u, respectively, and likewise when regarded as points of the ranges v, respectively

241. Problem *Between two given straight lines u and u_1 to pl a segment such that it shall subtend given angles at two given poi O and S* (Fig 170)

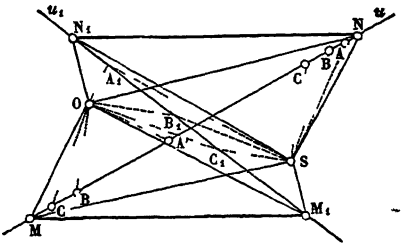


Fig 170

Draw any ray SA to meet u in A, draw SA_1 to meet u_1 in A_1 that ASA_1 may be equal to the second of the given angles, join O and draw OA' to meet u in A' so that $A'OA_1$ may be equal to t first of the given angles Then the problem would be solved if (coincided with OA' Three trials of a similar kind to the one just ma will give three pairs of corresponding rays (OA and OA', OB and O OC and OC') of the two projective pencils which would be traced by causing OA and OA' to change their positions simultaneous the self corresponding rays OM and ON of these pencils will give solutions (MM and NN) of the problem

242. PROBLEM *Given two projective ranges u and u', if a pair of corresponding points A and A' of these ranges be taken, it is required to find another pair of corresponding points M and M' such that the ratio of the length of the segment AM to that of the segment $A'M'$ may be equal to a given number λ.*

Let A and A', B and B', C and C' be three pairs of corresponding points of the two ranges. On u take two new points B'' C'' such that $AB''=\lambda A'B'$ and $AC''=\lambda A'C'$ The points A, B'', C'' determine a range which is similar (Art. 99) to the range A', B', C', and therefore projective with A, B, C, The collinear ranges A, B'', C'' and A, B, C, have already one self-corresponding point in A; their other self-corresponding point M (Art. 90) will give the solution of the problem, since $AM = AM'' = \lambda\ A'M'$ This problem is therefore of the first degree

243. PROBLEM. *Given two collinear projective ranges ABC and $A'B'C'$ ..., to find a pair of corresponding points M and M' such that the segment MM' shall be bisected at a given point O*

Take three points A'', B'', C'' such that O is the middle point of each of the segments AA'', BB'', CC'', the points A'', B'', C'' determine a range which is equal to the range ABC, and therefore projective ... nstruct the self-corresponding points of ... nges $A'B'C'$ and $A''B''C''$, if M' or ... MM' will have its middle point at O, and will be a segment such as is required

244. PROBLEM *Given a straight line and two points E, F on it, to determine on the straight line two points M and M' such that the segment MM' may be equal in length to a given segment, and the anharmonic ratio $(EFMM')$ equal to a given number*

Take on the given straight line any three points A, B, C, then find on it three points A', B', C' such that the anharmonic ratios $(EFAA')$, $(EFBB')$, $(EFCC')$ may each be equal to the given number, and again three points A'', B'', C'' such that the segments AA'', BB'', CC'' may each be equal in length to the given segment The ranges ABC and $A'B'C'$ will be projectively related (Arts 79, 109), and the same will be the case with regard to the ranges ABC and $A''B''C''$ (Art 103), therefore $A'B'C'$ and $A''B''C''$ will be projective with one another If these ranges have self-corresponding points, and if M' or M'' is one of them, the segment MM' and the anharmonic ratio $(EFMM')$ will have the given values, and the problem is solved

245 PROBLEM *To inscribe in a given triangle PQR a rectangle of given area* (Fig 171)

Suppose $MSTU$ to be the rectangle required, if MS' be drawn parallel to PR, a parallelogram $MSPS'$ will be formed which is equal

in area to the rectangle, so that for the given problem may be substituted the following equivalent one

To find on the base QR of a given triangle PQR a point M such that if MS, MS' be drawn parallel to the sides PQ, PR to meet PR, PQ in S, S' respectively, the rectangle contained by PS and PS' shall be equal to a given square k^2

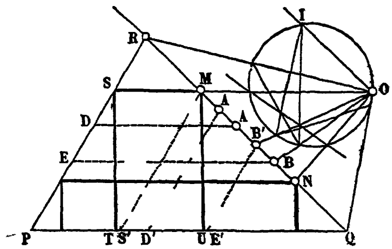


Fig 171

Take any point A on QR, draw AD parallel to PQ to meet PR in D, and take on PQ a point D' such that the rectangle contained by PD and PD' may be equal to k^2, then draw $D'A'$ parallel to PR to meet QR in A' If the points A and A' coincided with one another, the problem would be solved.

Now let the points A, D, D', A' be made to vary simultaneously; they will trace out ranges which are all projective with one another For since D is the projection of A made from the point at infinity on PQ, and A' the projection of D' made from the point at infinity on PR, the first and second ranges are in perspective, and the third and fourth likewise But the second and third ranges are projective with one another, since the relation $PD\ \ PD'=k^2$ shows (Art 74) that the points D and D', in moving simultaneously, describe two projective ranges such that the point P, regarded as belonging to either range, corresponds to the point at infinity regarded as belonging to the other *

Three similar trials give three pairs of points similar to A and A', if the self-corresponding points of the ranges determined by these pairs be constructed, they will give the solutions of the problem

Instead of taking the point A quite arbitrarily in the three trials, any particular positions may be chosen for it, and by this means the construction may often be simplified This remark applies to all the problems which we have discussed With regard to the present one, it is clear that if A be taken at infinity, its projection D will also lie at infinity, consequently D' will coincide with P, and therefore A' with R Again, if A be taken coincident with Q, its projection D will coincide with P, and consequently D', and therefore also A', will pass off to infinity We have thus two trials, neither of which requires

* If the two ranges be called u and u, and the construction of Art 85 (left) be referred back to, it will be seen that the auxiliary range u'' lies in this case entirely at infinity If then a pair of corresponding points D and D' have been found, and we wish to find the point E which corresponds to any other point E of PR ($\equiv u$), we have only to join $D'E$, and to draw DE' parallel to $D\ E$ to meet PQ ($\equiv u'$) in E'

any construction, the pairs which result from them are composed respectively of the point at infinity and R, and of Q and the point at infinity If the pair given by the third trial be called B, B', and if A, A' stand for any pair whatever, we have (Art 74)

$$QA \quad RA' = QB \quad RB',$$

and therefore, if M is a self-corresponding point,

$$QM \quad RM = QB \quad RB',$$

from which the self-corresponding points could be found But it is better in all cases to go back to the general construction of Art 206 In this case the three pairs of conjugate points of the two ranges which are given are B and B', the point at infinity and R, Q and the point at infinity Let then any circle be taken, and a point O on its circumference, from O draw the straight lines OB, OB', OR, OQ, and a parallel to QR, and let these cut the circle again in B_1, B_1', R_1, Q_1, and I respectively * Join the point of intersection of $B_1 R_1$ and $B_1' I$ with that of $B_1 I$ and $B_1' Q_1$, if the joining line cut the circle in two points M_1 and N_1, the straight lines which join these to O will meet QR in the self-corresponding points M and N, and these give the solutions of the problem

246 PROBLEM *To construct a polygon, whose sides shall pass ugh given points, and all whose vertices except one shall y on given straight lines, and which shall be such that the angle included by the sides which meet in the last vertex is equal to a given angle*

Suppose, for example, that it is required to construct a triangle LMN (Fig 172) whose sides MN, NL, LM shall pass through the given points O, V, U respectively, and whose vertices M, N shall lie on the given straight lines u, v respectively, and which shall be such that the angle MLN is equal to a given angle

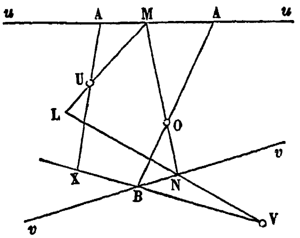


Fig 172

Through O draw any straight line to cut u in A and v in B, join BV, and through U draw the straight line UX making with BV an angle equal to the given one Let UX meet u in A', the problem would be solved if the point A' coincided with A If the rays OA, UA' be made to vary simultaneously, they will determine on u two projective ranges, the solutions of the problem will be found by constructing the self corresponding points of these ranges

* Of these points only I is marked in the figure

247 The following problem is included in the foregoing one

A ray of light emanating from a given point O is reflected from n given straight lines in succession, to determine the original direction which the ray must have, in order that this may make with its direction after the last reflexion a given angle

Let $u_1, u_2, \ldots u_n$ be the given straight lines (Fig 173) If the ray OA_1 strike u_1 at A_1, then by the law of reflexion the incident and reflected rays will make equal angles with u_1, but the incident ray passes through the fixed point O, therefore the reflected ray will always pass through the point O_1 which is symmetrical to O with regard to u_1* So again, if the ray after one reflexion strikes u_2 at A_2, it will be reflected according to

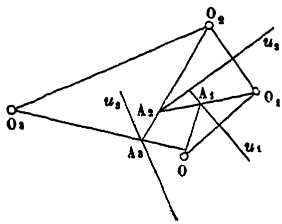


Fig 173

the same law, consequently the ray after two reflexions will pass through a fixed point O_2 which is symmetrical to O_1 with regard to u_2, and so on The paths of the ray before reflexion, and after one, two, $\ldots n$ reflexions form therefore a polygon $OA_1A_2A_3\ldots$, whose $n+1$ sides pass respectively through $n+1$ fixed points $O, O_1, O_2, \ldots O_n$, and which is such that n of its vertices lie respectively on n given straight lines $u_1, u_2, \ldots u_n$, while the angle included by the sides which meet in the last vertex is to be equal to a given angle Thus the problem reduces, as was stated, to that of Art 246

248 PROBLEM *To construct a polygon whose vertices shall lie respectively on given straight lines, and whose sides shall subtend given angles at given points respectively*

Suppose it required to construct a triangle whose vertices 1, 2, 3 shall lie on the given straight lines u_1, u_2, u_3 respectively, and whose sides 23, 31, 12 shall subtend at the given points S_1, S_2, S_3 respectively the angles $\omega_1, \omega_2, \omega_3$ which are given in sign and magnitude (Fig 174) On u_1 take any point A, join AS_3, and make the angle AS_3B equal to ω_3, let S_3B cut u_2 in B Join BS_1, make the angle BS_1C equal to ω_1, and let S_1C cut u_3 in C Join CS_2, make the angle CS_2A' equal to ω_2, and let S_2A' cut u_1 in A' The problem would be solved if S_2A' coincided

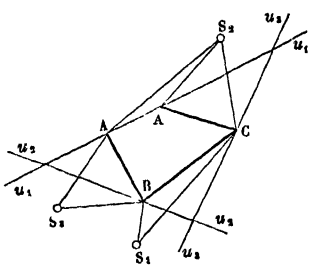


Fig 174

* i e a point O_1 such that OO_1 is bisected at right angles by u_1

with S_2A. If S_2A be made to turn about S_2, the other rays S_3A, S_3B, S_1B, S_1C, S_2C, and S_2A' will change their positions simultaneously, and will trace out pencils which are all projectively related. For the ranges traced out by S_3A and S_3B respectively will be projective (Art. 108) since the angle AS_3B is constant; the ranges traced out by S_3B and S_1B respectively are projective since they are in perspective; and so on. The solutions of the problem will therefore be given by the self-corresponding rays of the concentric projective pencils which are generated by S_2A and S_2A' respectively.

In the same manner is solved the more general problem in which the straight lines joining S_1, S_2, ... to the vertices of the polygon are no longer to include given angles, but are to be such that together with pairs of given straight lines meeting in S_1, S_2, ... respectively they form at each of these points a pencil of four rays having a given anharmonic ratio. If at each of the points the pencil is to be harmonic, and the given straight lines such as to include a right angle, the problem can be enunciated as follows (Art. 60):

To construct a polygon whose vertices shall lie respectively on given straight lines, and whose sides shall subtend at given points angles whose bisectors are given.

249 The same method gives the solution of the problem

To construct a polygon whose sides shall pass respectively through [illegible] *e such that the pairs of adjacent sides* [illegible] *ments respectively in given anharmonic ratios.* *

Particular cases of this problem may be obtained by supposing that each pair of adjacent sides is to intercept on a given straight line a segment given in magnitude and direction; or a segment which is divided by a given point into two parts having a given ratio to one another.†

* That is to say, two adjacent sides are to cut a given straight line, on which are two given points A, B, in two other points C, D such that the anharmonic ratio $(ABCD)$ may be equal to a given number.

† CHASLES, *Géom. sup.*, pp. 219–223, and TOWNSEND, *Modern Geometry* (Dublin, 1865), vol. ii. pp. 257–275.

CHAPTER XX.

POLE AND POLAR

250 LET any point S be taken in the plane of a conic (Fig 175), and through it let any number of transversals be drawn to cut the conic in pairs of points A and A', B and B', C and C', The tangents a and a', b and b', c and c' at these points will, by Arts 203, 204, intersect in pairs on a fixed straight line s, on which lie also the points of contact of the tangents from S to the conic (when the position of S is such that tangents can be drawn) Further, the pairs of chords AB' and $A'B$, AC' and $A'C$, BC' and $B'C$, AB and $A'B'$, AC and $A'C'$, BC and $B'C'$, will intersect on s Another property of the straight line s may be noticed In the complete quadrangle $AA'BB'$, each of the straight lines AA' and BB' is divided harmonically by the diagonal point S and the point where it is cut by the straight line s which joins the other diagonal points (Art 57), consequently A and A' (and similarly B and B', C and C',) are harmonic conjugates with regard to S and the point where AA' (or BB', CC',) is cut by s

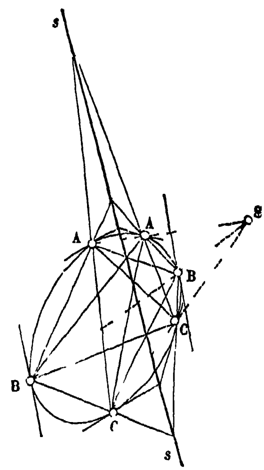


Fig 175

The straight line s determined in this manner by the point S is called the *polar* of S with respect to the conic, and, reciprocally, the point S is said to be the *pole* of the straight line s

The polar of a given point S is therefore at the same time (1) *the*

locus of the points of intersection of tangents to the conic at the pairs of points where it is cut by any transversal through S, (2) the locus of the points of intersection of pairs of opposite sides of quadrangles inscribed in the conic such that their diagonals meet in S, (3) the locus of points taken on any transversal through S such that they are harmonically conjugate to S with regard to the pair of points in which the transversal is cut by the conic, (4) the chord of contact of the tangents from S to the conic, when S has such a position that it is possible to draw these * †

251. Reciprocally, any given straight line s determines a point S, of which it is the *polar* For let A and B (Fig 176) be any two points on the conic, the tangents a and b at these points will cut s in two points from which can be drawn two other tangents a' and b' to the conic Let A' and B' be the points of contact of these, and let AA', BB' meet in S, then the polar of S will pass through the points aa' and bb', and must therefore coincide with s

If then from any point on s a pair of tangents can be drawn to the conic, their chord of contact will pass through S

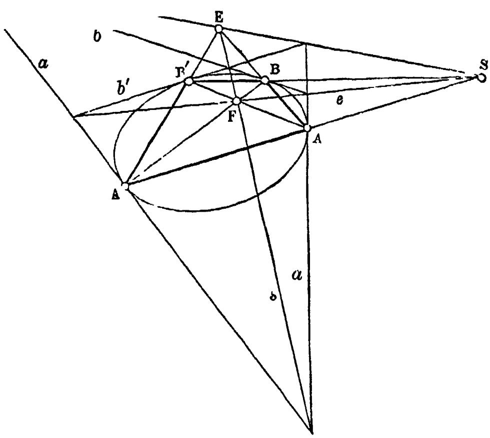


Fig 176

252 The complete quadrangle $AA'BB'$ and the complete quadrilateral $aa'bb'$ (Fig 176) have the same diagonal

* APOLLONIUS, *loc cit*, lib VII 37, DESARGUES, *loc cit*, pp 164 sqq, DE LA HIRE, *loc cit*, books I and II

† (4) follows from (3) by what has been proved in Art 71

triangle (Art 169) The vertices of this triangle are S, the point of intersection F of AB and $A'B'$, and the point of intersection E of AB' and $A'B$, its sides are s, the straight line f joining the points ab and $a'b'$, and the straight line e joining the points ab' and $a'b$ Thus *if from any two points taken on the straight line s pairs of tangents a and a', b and b' be drawn to the conic, the diagonals of the quadrilateral $aba'b'$ will pass through S*

253 The straight lines a, a', b, b' (Fig 177) form a quadrilateral circumscribed about the conic, one of whose diagonals is s, and whose other two diagonals meet in S *Thus if from any point on s a pair of tangents be drawn to the conic, they will be harmonically conjugate with regard to s and the straight line joining the point to S* (Art 56)

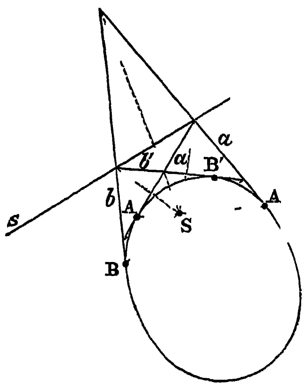


Fig 177

254 *If then a conic is given, every point in its plane has its polar and every straight line has its pole** The given conic, with reference to which the pole and polar are considered, may be called the *auxiliary* conic

I If a point in the plane of a conic is such that from it two tangents can be drawn to the curve, it is said to lie *outside* the conic, or to be an *external* point, if it is such that no tangent can be drawn, it is said to lie *inside* the conic, or to be an *internal* point If then the pole lies outside the conic (Art 203, VIII) the polar cuts the curve, and it cuts it at the points of contact of the tangents from the pole to the conic †

If the pole lies inside the curve, the polar does not cut the conic

II If a point on the conic itself be taken as pole and a transversal be made to revolve round this point, one of its points of intersection with the conic will always coincide with the pole itself Since then the polar is the locus of the points where the tangents at these points of intersection meet, and

* DESARGUES, *loc cit*, p 190

† See also Art 250, (4)

in this case one of the tangents is fixed, it follows that the polar of a point on the conic is the tangent at this point, or that *if the pole is a point on the conic, the polar is the tangent at this point*

III. Reciprocally, if every point of the polar lies outside the conic, the pole lies inside the conic, if the polar cuts the conic, the pole is the point where the tangents at the two points of intersection meet, and if the polar touches the conic, the pole is its point of contact

255 *If two points are such that the first lies on the polar of the second, then will also the second lie on the polar of the first*

Consider Fig. 176; let E be taken as pole and let F be a point lying on the polar of E If the straight line EF cuts the conic, it will cut it in two points which are harmonically conjugate with regard to E and F (Art. 250 [3]), consequently one of the points E, F will lie inside and the other outside the conic, and by Art 250 (3) again, if F be taken as pole, E will be a point on its polar

If the straight line EF does not cut the conic, the chord of contact of the tangents from E will pass through F, since this chord is the polar of E and therefore by Art 250 (1) E will

The above proposition may also be expressed in the following manner

If a straight line f pass through the pole of another straight line e, then will also e pass through the pole of f

For let E, F be the poles of e, f respectively, since by hypothesis E lies on the polar of F, therefore F will lie on the polar of E that is to say, e will pass through F, the pole of f

Two points such as E and F, which possess the property that each lies on the polar of the other, are termed *conjugate* or *reciprocal* points with respect to the conic And two straight lines such as e and f, each of which passes through the pole of the other, are termed *conjugate* or *reciprocal* lines with respect to the conic

The foregoing proposition may then be enunciated as follows

If two points are conjugate to one another with respect to a conic, their polars also are conjugate to one another, and conversely

256 The same proposition can be put into yet another form, viz

Every point on the polar of a given point E has for its polar a straight line passing through E

Every straight line passing through the pole of a given straight line e has for its pole a point lying on e *

In other words, if a variable pole F be supposed to describe a given straight line e, the polar of F will always pass through a fixed point E, the pole of the given line, and conversely, if a straight line f revolve round a fixed point E, the pole of f will describe a straight line e, the polar of the given point E

Or again *the pole of a given straight line e is the centre of the pencil formed by the polars of all points on e, and the polar of a given point E is the locus of the poles of all straight lines passing through E* †

257 PROBLEM *Given a point S, to construct its polar with respect to a given conic*

I Let the conic be determined by five points A, B, C, D, E (Fig 178)

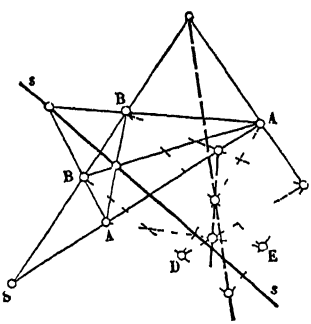


Fig 178

Join SA, SB, and find the points A', B' where these cut the conic again respectively (Art 161, right) The straight line s which joins the point of intersection of AB' and $A'B$ to that of AB and

Given a straight line s, to construct its pole with respect to a given conic

I Let the conic be determined by five tangents a, b, c, d, e (Fig 179)

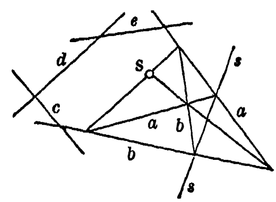


Fig 179

From the points sa, sb draw the second tangents a', b' respectively to the conic (Art 161, left) The point S in which the diagonals of the quadrangle $aba'b'$ intersect one another will be

* DESARGUES, *loc cit*, p 191
† PONCELET, *loc cit*, Art 195

$A'B'$ will be the polar of the given point (Art 250 [2])

II Let the conic be determined by five tangents a, b, c, d, e (Fig 180).

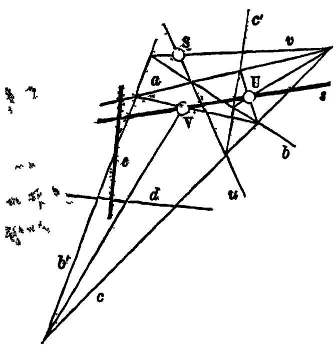


Fig 180

- t e , will ..6) To ... transversal u may be drawn through the point ab, if then the second tangent c' be drawn to the conic (Art 161) from the point uc, U will be the point of intersection of the diagonals of the quadrilateral $acbc'$ So too if the transversal v be drawn through the point ac for example, and the second tangent b' be drawn to the conic from the point vb, then V will be the point of intersection of the diagonals of the quadrilateral $abcb'$

the pole of the given straight line

II Let the conic be determined by five points A, B, C, D, E (Fig 181)

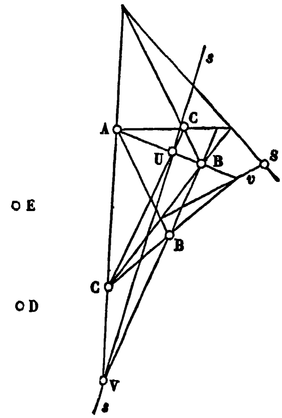


Fig 181

On s take two points U and V, and construct their polars u and v (as on the left hand side above), the point uv will be the pole of s (Art 256) To simplify matters the point U may be taken on the straight line AB, if then UC be joined, and the second point C' in which it meets the conic be constructed, u will be the straight line joining the points of intersection of the pairs of opposite sides of the quadrangle $ACBC'$ So too if V be taken on the straight line AC for example, and VB be joined, and its second point of intersection B' with the conic be constructed, then v will be the straight line joining the points of intersection of the pairs of opposite sides of the quadrangle $ABCB'$

258 Let E and F (Fig 182) be a pair of conjugate points

and let G be the pole of EF, then G will be conjugate both to E and to F, so that the three points E, F, G are conjugate to one another two and two Every side therefore of the triangle EFG is the pole of the opposite vertex, and the three sides are conjugate lines two and two

A triangle such as EFG, in which each vertex is the pole of the opposite side with regard to a given conic is called a *self-conjugate* or *self-polar* triangle with regard to the conic

Self polar triangle

259 *To construct a triangle self-conjugate with regard to a given conic*

One vertex E (Fig 182) may be taken arbitrarily, construct its

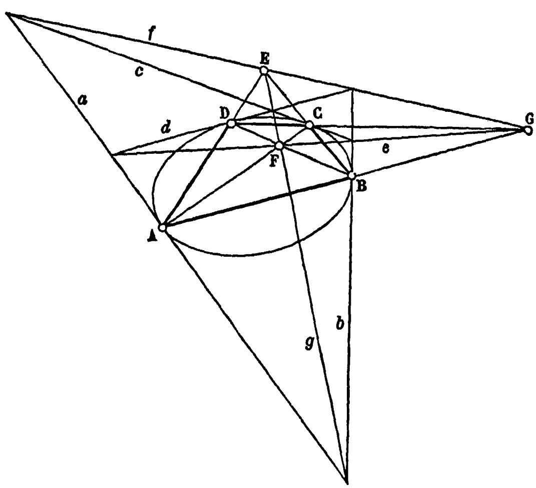


Fig 182

polar, take on this polar any point F, and construct the polar of F This last will pass through E, since E and F are conjugate points, if G be the point where it cuts the polar of E, then E and G, F and G, will be pairs of conjugate points, and therefore EFG is a self-conjugate triangle

In other words take any point E and draw through it any two transversals to cut the conic in A and D, B and C respectively, join AC, BD, meeting in F, and AB, CD meeting in G, then EFG is a self conjugate triangle

Or again, one side e may be taken arbitrarily, and its pole E constructed, if through E any straight line f be drawn, and its pole

(which will lie on e) be constructed and joined to the pole of e by the straight line g, then efg will be a triangle such as is required, for the straight lines e, f, g are conjugate two and two

Thus, after having taken the side e arbitrarily, we may proceed as follows take two points on e and from them draw pairs of tangents a and d, b and c, to the conic, join the points ac, bd by the straight line f, and the points ab, cd by the straight line g, then will efg be a self-conjugate triangle

260 From what has been said above the following property is evident

*The diagonal points of the complete quadrangle formed by any four points on a conic are the vertices of a triangle which is self-conjugate with regard to the conic And the diagonals of the complete quadrilateral formed by any four tangents to a conic are the sides of a triangle which is self-conjugate with regard to the conic**

Or, in other words

The triangle whose vertices are the diagonal points of a complete quadrangle is self-conjugate with regard to any conic circumscribing the quadrangle And the triangle whose sides are the diagonals of a ... is self-conjugate with regard to any conic ... e quadrilateral

261 From the properties of the circumscribed quadrilateral and the inscribed quadrangle (Arts 166 to 172) it follows moreover that

If EFG (Fig 182) is a triangle self-conjugate with regard to a given conic, and ABC is a triangle inscribed in the conic, such that two of its sides CA, AB pass through two of the vertices F, G respectively of the other triangle, then will the remaining side BC pass through the remaining vertex E, and every side of the inscribed triangle will be divided harmonically by the corresponding vertex of the self-conjugate triangle and the side which joins the other two vertices of it

The three straight lines EA, FB, GC meet in one point D on the conic, the two triangles are therefore in perspective, and the three pairs of corresponding sides FG and BC, GE and CA, EF and AB, will meet in three collinear points

Hence it follows that a self-conjugate triangle EFG and a point A of a conic determine an inscribed quadrangle $ABCD$, whose diagonal

* DESARGUES, *loc cit*, p 186

triangle is EFG. The points B, C, D are those in which the straight lines AG, AF, AE cut the conic again.

The enunciation of the correlative property is left to the student.*

262. Of the three vertices of the triangle EFG, one always lies inside the conic, and the two others outside it. For if E is an internal point, its polar does not cut the conic, and consequently F and G are both external to the conic. If, on the other hand, E is an external point, its polar cuts the conic, and F and G are harmonic conjugates with regard to the two points of intersection; of the two points F and G therefore, one must be internal and the other external to the conic.

From this property and that of Art. 254, I, we conclude that of the three sides of any self-conjugate triangle, two always cut the curve, and the third does not.

263. (1) *On every straight line there are an infinite number of pairs of points which are conjugate to one another with respect to a given conic, and these form an involution.*†

(2) *Through every point pass an infinite number of pairs of straight lines which are conjugate to one another with respect to a given conic, and these form an involution.*†

(3) *If a point describes a range, its polar with respect to a given conic will trace out a pencil which is projective with the given range. And, conversely, if a straight line describes a pencil, its pole with respect to a given conic will trace out a range which is projective with the given pencil.*‡

To prove these theorems, consider Fig. 183, and suppose in it the conic and the three points A, B, G to be given. Let the point C be supposed to move along the conic. Then the rays AC, BC will trace out two pencils which are projective with one another (Art. 149 [1]), and therefore the ranges in which these pencils cut the polar of G will be projective also; that is to say, the conjugate points F and E will describe two collinear projective ranges. In these ranges the points F and E correspond to one another doubly, since the polar of E passes through F, and the polar of F passes through E; consequently the ranges in question are in involution.

From what has been said it follows also that the pairs of

* PONCELET, *loc. cit.*, p. 104. † DESARGUES, *loc. cit.*, pp. 192, 193.
‡ MÖBIUS, *Baryc. Calc.*, § 290.

conjugate lines GF, GE in like manner form an involution, and that the range of poles E, F, is projective with the pencil of polars GF, GE,

264 If the straight line EF cuts the conic, the two points of

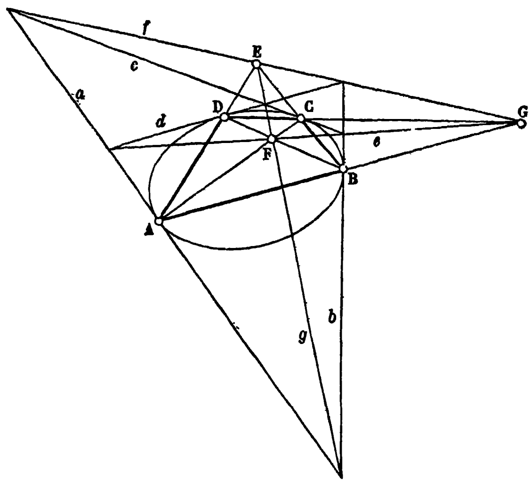


Fig 183

section are the double points of the involution formed by pairs of conjugate poles The centre of the involution lies on the diameter which passes through the pole G of the given straight line (Art 290)

If the point G is external to the conic, the tangents from G to the conic are the double rays of the involution formed by the pairs of conjugate polars

Consequently (Art 125)

A chord of a conic is harmonically divided by any pair of points lying on it which are conjugate with respect to the conic, and

The pair of tangents drawn from any point to a conic are harmonic conjugates with respect to any pair of straight lines meeting in the given point which are conjugate with respect to the conic

If the point G lies at infinity, the pairs of conjugate straight lines form an involution of parallel rays, the central ray of which is a diameter of the conic (Arts 129, 276)

265 THEOREM *If two complete quadrangles have the same diagonal points, their eight vertices lie either four and four on two straight lines or else they all lie on a conic*

Let $ABCD$ and $A'B'C'D'$ (Fig 184) be two quadrangles having the same diagonal points E, F, G, so that

$$\begin{array}{llll} BC, AD, B'C', A'D' & \text{all meet in} & & E, \\ CA, BD, C'A', B'D' & \quad ,, & \quad ,, & F, \\ AB, CD, A'B', C'D' & \quad ,, & \quad ,, & G \end{array}$$

(1) In the first place let the eight vertices be such that some three of them are collinear Suppose for example that A' lies on AB Since AB and $A'B'$ meet in G, therefore B' also must lie on AB, and since the straight lines GE, GF are harmonically conjugate with regard both to AB, CD and to $A'B', C'D'$, and AB coincides with $A'B'$, therefore also CD coincides with $C'D'$ Thus the four points C, D, C', D' are collinear, and the eight points $A, B, C, D, A', B', C', D'$ lie four and four on two straight lines

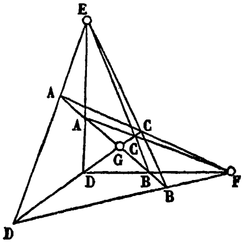


Fig 184

(2) But if this case be excluded, *i e* if no three of the eight vertices lie in a straight line, then a conic can be drawn through any five of them Let a conic be drawn through A, B, C, D, A' (Fig 185), then shall B', C', D' lie on the same conic For since E, F, G are the diagonal points of the inscribed quadrangle $ABCD$, G is the pole of EF, and therefore G and the point where its polar EF meets the transversal $GB'A'$ are harmonically conjugate with regard to the points where this transversal cuts the conic But one of these last points is A', therefore the other is B', for since E, F, G are also the diagonal points of the quadrangle $A'B'C'D'$, the points A' and B' are harmonically conjugate with regard to G and the point where EF cuts $A'B'$ In a similar manner it can be shown that C' and D' also lie on the same conic The eight vertices $A, B, C, D, A', B', C', D'$ therefore lie on a conic, and the proposition is proved

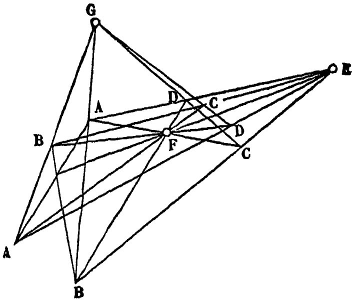


Fig 185

Since the straight lines AB and $A'B'$ meet in G, therefore AA' and BB', as also AB' and $A'B$, will meet on EF, the polar of G This property gives the means of constructing the point B' when the points A, B, C, D, A' are given The point C' will then be found as the point of intersection of $A'F$ and $B'E$, and the point D' as that of $B'F, A'E$, and $C'G$

266 Suppose now that two conics are given which are inscribed in the same quadrilateral Let the four common tangents which form this quadrilateral be a, b, c, d, and let their points of contact with the conics be A, B, C, D and A', B', C', D' respectively By the theorem of Art 169, the triangle formed by the diagonals of the circumscribed quadrilateral $abcd$ has for its vertices the diagonal points of the inscribed quadrangle $ABCD$ and also those of the inscribed quadrangle $A'B'C'D'$, thus $ABCD$ and $A'B'C'D'$ have the same diagonal points Accordingly, by the theorem of Art 265, *the eight points* $A, B, C, D, A', B', C', D'$ *lie either four and four on two straight lines, or they lie all on a conic*

267 By writing, as usual, line for point, and point for line, the propositions correlative to those of Arts 265 and 266 can

If two complete quadrilaterals have the same three diagonals, their eight sides either pass four and four through two points, or else they all touch a conic

If two conics intersect in four points, the eight tangents to them at these points either pass four and four through two points, or they all touch a conic *

take

268 If there be given the diagonal points E, F, G and one vertex A of a quadrangle $ABCD$, the quadrangle is completely determined, and can be constructed For D is that point on AE which is harmonically conjugate to A with respect to E and the point where FG cuts AE, so C is that point on AF which is harmonically conjugate to A with respect to F and the point where GE cuts AF, and B is that point on AG which is harmonically conjugate to A with respect to G and the point where EF cuts AG

But if there be given the diagonal points E, F, G of a quadrangle $ABCD$ and the conic with respect to which EFG is a self-conjugate triangle, the quadrangle is not completely

* Staudt, *loc cit*, p 293

determined For we may take arbitrarily on the conic a point A as one vertex of the quadrangle $ABCD$, then the other vertices B, C, D are the second points of intersection of the conic with the straight lines AG, AF, AE respectively Hence it follows that

All conics with respect to which a given triangle EFG is self-conjugate, and which pass through a fixed point A, pass also through three other fixed points B, C, D

269 PROBLEM *To construct a conic passing through two given points A and A', and with respect to which a given triangle EFG shall be self-conjugate.*

Solution Construct, in the manner just shown, the three points B, C, D which form with A a complete quadrangle having E, F, and G for its diagonal points Five points A, A', B, C, D on the conic are then known, and by means of Pascal's theorem any number of other points on it may be found Or we may construct the three points B', C', D' which form with A' a complete quadrangle having E, F, and G for its diagonal points, the eight points A, B, C, D, A', B', C', D' will then all lie on the conic required

270 Consider again the problem (Art 218) of describing a conic to touch four given straight lines a, b, c, d and to pass through a given point S (Fig 186) The diagonals of the quadrilateral $abcd$ form a triangle EFG which is self-conjugate with regard to the conic, consequently, if the three points P, Q, R be constructed which together with S form a quadrangle having E, F, and G for its diagonal points, the three points so constructed will lie also on the required conic Now it may happen that there is no conic which satisfies the problem, or

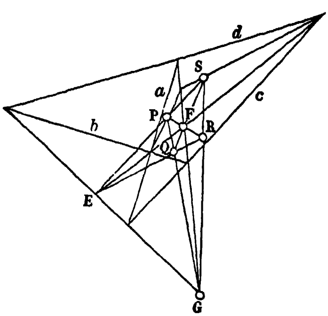


Fig 186

again there may be two conics which satisfy it (Art 218, right), in the second case, since the construction for the points P, Q, R is linear, the two conics will both pass through these points Thus

If two conics inscribed in the same quadrilateral abcd pass through the same point S, they will intersect in three other points P, Q, R, and the triangle formed by the diagonals of the circumscribed quadrilateral abcd will coincide with that formed by the diagonal points of the inscribed quadrangle $PQRS$

In order to find a construction for the points P, Q, R, consider

the point P for example which lies on ES (Fig 186) It is seen that the segment SP must be divided harmonically by E and its polar FG (Art 250), but the diagonal (ab) (cd) which passes through E is also divided harmonically, at E and F We have therefore two harmonic ranges, which are of course projective (Art 51) and which are in perspective since they have a self-corresponding point at E, therefore the straight lines P (ab), S (cd), and FG, which join the other pairs of corresponding points, will meet in a point (Art 80) We must therefore join S to one extremity of one of the diagonals passing through E, for example to the point cd, and take the point where the joining line meets FG This point, when joined to the other extremity ab of the diagonal, will give a straight line which will meet ES in the required point P *

271. The propositions and constructions correlative to those of the last three Articles, and which will form useful exercises for the student, are the following

All conics with respect to which a given triangle is self conjugate, and which touch a fixed straight line, touch three other fixed straight lines

To construct a conic to touch two given straight lines, and with respect to which a given triangle shall be self-conjugate

If two conics circumscribing the same quadrangle have a common tangent, they have three other common tangents

hree remaining common tangents to two conics which pass through four given points and touch a given straight line (Art 218, left)

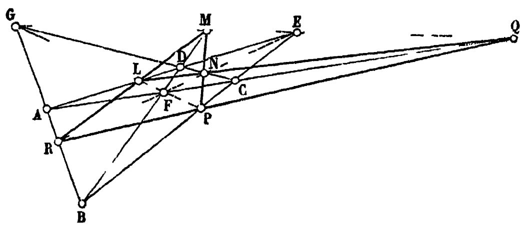


Fig 187

272 Let $ABCD$ (Fig 187) be a complete quadri~~lateral~~ whose diagonal points are E, F, and G Let also

L and P be the points where FG meets AD and BC respectively
M and Q „ „ GE „ BD and CA „
N and R „ „ EF „ CD and AB „

The six points so obtained are the vertices of a complete quadrilateral For the triangle EFG is in perspective with each of the

* BRIANCHON *loc cit*, p 45, MACLAURIN, *De lin Geom*, § 43

triangles ABC, DCB, CDA, BAD, the centres of perspective being D, A, B, C respectively, whence it follows that the four triads of points PQR, PMN, LQN, and LMR lie on four straight lines (the axes of perspective)

These four axes form a quadrilateral whose diagonals LP, MQ, NR form the triangle EFG Accordingly, a conic inscribed in the quadrangle $ABCD$ and passing through L will pass also through N, P, and R (Art 270), similarly a conic can be inscribed in the quadrangle $ABDC$ to pass through R, M, N, and Q, and a conic can be inscribed in the quadrangle $ACBD$ to pass through Q, P, M, and L

It will be seen that for each of these conics the four tangents shown in the figure (the four sides of the complete quadrangle $ABCD$) are harmonic, and that the same will therefore be the case with regard to their points of contact (Arts 148, 204) For take one of the sides of the quadrangle, for example AB, a consideration of the complete quadrangle $CDEF$ shows that this side is harmonically divided in R and G Now the points A, B, G are the points of intersection of the tangent AB with the other three tangents, and R is the point of contact of AB, therefore the four tangents are cut by any other tangent to the conic in four harmonic points *

273 If $ABCD$ is a parallelogram, the points E, G, M, Q pass off to infinity, and $LNPR$ also becomes a parallelogram Of the three conics considered above the first will in this case be an ellipse which touches the sides of the parallelogram $ABCD$ at their middle points, the second a hyperbola which touches the sides AB and CD at their middle points and has AC and BD for asymptotes, and the third a hyperbola having the same asymptotes and touching the sides AD and BC at their middle points

274 From that corollary to Brianchon's theorem which has reference to a quadrilateral circumscribed about a conic (Art 172) we have already, in Art 173, deduced a method for the construction of tangents to a conic when we are given three tangents a, b, c and the points of contact B, C of two of them (Fig 183) We take any point E on BC and join it to the points ab, ac by the straight lines g, f, respectively, if the point in which g meets c be joined to that in which f meets b, the joining line d will be a tangent to the conic

The four tangents a, b, c, d form a complete quadrilateral two of whose diagonals $g \equiv (ab)(cd)$ and $f \equiv (ac)(bd)$ intersect

* STEINER *loc cit*, p 160, § 43, 4, Collected Works, vol 1 p 347, STAUDT, *Beitrage zur Geometrie der Lage*, Art 329

in E, therefore also (Art 172) the chords of contact AD and BC of the tangents a and d, b and c respectively will intersect in E The straight lines joining E to the points ab and ac, being two of the diagonals of the quadrilateral $abcd$, are conjugate lines with respect to the conic, consequently

If a triangle abc is circumscribed about a conic, the straight lines which join two of its vertices ab and ac to any point E on the polar of the third vertex bc are conjugate to one another with respect to the conic

And conversely

If two straight lines (c and b) touch a conic, any two conjugate straight lines (f and g) drawn from any point (E) on their chord of contact will cut the two given tangents in points such that the straight line (a) joining them touches the conic

275 Let us now investigate the correlative property Suppose three points A, B, C on a conic to be given, and the tangents b, c at two of these points (Fig 183) If a straight line e drawn arbitrarily through the point bc cut AB in G and AC in F, then if GC and FB be joined they will intersect in oint D lying on the conic

The four points A, B, D, C form a complete quadrangle two of whose diagonal points lie on e, therefore (Art 166) the point bc and the point of intersection of the tangents at A and D will lie on e The points G and F, being two of the diagonal points of the quadrilateral $ABCD$, are conjugate with respect to the conic, consequently

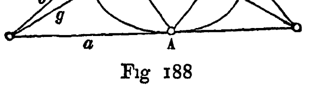


Fig 188

If a triangle ABC (Fig 188) is inscribed in a conic, the points F and G in which two of the sides are cut by any straight line drawn through the pole S of the third side are conjugate to one another with respect to the conic

And conversely

If two given points (B, C) on a conic be joined to two conjugate points (G, F) which are collinear with the pole (S) of the chord (BC) joining the given points, then the joining lines will intersect in a point (A) lying on the conic

CHAPTER XXI

THE CENTRE AND DIAMETERS OF A CONIC

276 LET an infinitely distant point be taken as pole, and through it let a transversal be drawn (Fig 189) to cut the conic in two points A and A' The segment AA' will be harmonically divided by the pole and the point where it is cut by the polar (Art 250), this point will therefore be the middle point of AA' (Art 59) That is to say

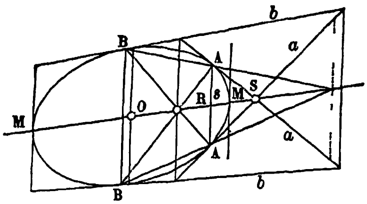


Fig 189

If any number of parallel chords of a conic be drawn, the locus of their middle points is a straight line, and this straight line is the polar of the point at infinity in which the chords intersect *

277 This straight line is termed the *diameter* of the chords which it bisects If the diameter meets the conic in two points, these will be the points of contact of the tangents drawn to the conic from the pole, i e of those tangents which are parallel to the bisected chords If the tangents at the extremities A and A' of one of these chords be drawn, they will meet in a point on the diameter If AA' and BB' are two of the bisected chords, the straight lines AB and $A'B'$, AB' and $A'B$ will intersect in pairs on the diameter (Art 250)

If, conversely, from a point on the diameter can be drawn a pair of tangents a and a' to the conic, their chord of contact AA' will be bisected by the diameter, and if through the same point there be drawn the straight line which is harmonically conjugate to the diameter with respect to the two

* APOLLONIUS, *Conic*, lib i 46, 47, 48, lib ii 5 6, 7, 28–31, 34–37

tangents, this straight line will be parallel to the bisected chords If from two points on the diameter there be drawn two pairs of tangents a and a', b and b', the straight line joining the points ab and $a'b'$ and that joining the points ab' and $a'b$ will both be parallel to the bisected chords (Art 252)

278 To each point at infinity, that is, to each pencil of parallel rays, corresponds a diameter The diameters all pass through one point, for they are the polars of points lying on one straight line, *viz* the straight line at infinity, the point in which the diameters intersect is the pole of the straight line at infinity (Art 256)

279 Since every parabola is touched by the straight line at infinity, and the point of contact is the pole of this straight line (Art 254, II), it follows (Art 278) that all diameters of a parabola are parallel to one another (they all pass through the point at infinity on the curve), and conversely, every straight line which cuts a parabola at infinity is a diameter of the curve

280 If S is any point from which a pair of tangents a and a' can be drawn to the conic (Fig. 189), the chord of contact AA', the polar of S, will be bisected at R by the diameter which passes through S, for S and the point at infinity on AA' are conjugate points with respect to the conic If the diameter cuts the curve in M and M', the tangents at these points are parallel to AA', and MM' is divided harmonically by the pole S and the polar AA' (Art 250)

If then the conic is a parabola (Fig 190) the point M' moves off to infinity, and therefore M is the middle point of the segment SR, thus

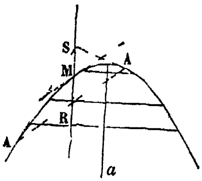


Fig 190

The straight line which joins the middle point of a chord of a parabola to the pole of the chord is bisected by the curve *

281 When the conic is not a parabola, the straight line at infinity is no longer a tangent to the curve, and consequently the pole of this straight line, or the point of intersection of the diameters, is a point lying at a finite distance Since any two points on the conic which are collinear with the pole are separated harmonically by the pole and the polar (Art 250), the pole will lie midway between the two points on the curve

* APOLLONIUS *loc cit*, lib I 35

when the polar lies at infinity. Every chord of the conic therefore which passes through the pole of the straight line at infinity is bisected at this point

On account of this property the pole of the straight line at infinity or the point in which all the diameters intersect is called the *centre* of the conic

282 Applying the properties of poles and polars in general (Arts 250—253) to the case of the centre and the straight line at infinity, it is seen (Fig 191) that

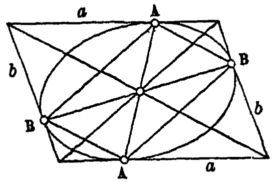


Fig 191

If A and A' are any pair of points on the conic collinear with the centre, the tangents at A and A' are parallel

If A and A', B and B' are any two pairs of points on the conic which are collinear with the centre, the pairs of chords AB and $A'B'$, AB' and $A'B$ are parallel, so that the figure $ABA'B'$ is a parallelogram

If a and a' are any pair of parallel tangents, their chord of contact passes through the centre, as also does the straight line lying midway between a and a' and parallel to both

If a and a', b and b' are any two pairs of parallel tangents, the straight line joining the points ab and $a'b'$ and that joining the points ab' and $a'b$ both pass through the centre, in other words, if $aba'b'$ is a parallelogram circumscribed to the conic, its diagonals intersect in the centre

283. If the conic is a hyperbola, the straight line at infinity cuts the curve, consequently the centre is a point exterior to the curve (Art 254, I) in which intersect the tangents at the infinitely distant points, *i e* the asymptotes (Fig 197)

If the conic is an ellipse, the straight line at infinity does not cut the curve, consequently the centre is a point inside the curve (Figs 191, 192)

284 Two diameters of a central conic (ellipse or hyperbola *) are termed *conjugate* when they are conjugate straight

* In the case of the parabola there are no pairs of conjugate diameters, for since the centre lies at infinity, the diameter drawn parallel to the chords which are bisected by a given diameter must coincide always with the straight line at infinity

lines with respect to the conic, i e when each passes through the pole of the other (Art 255)

Since the pole of a diameter is the point at infinity on any of the chords which the diameter bisects, it follows that the diameter b' conjugate to a given diameter b is parallel to the chords bisected by b, conversely, b' bisects the chords which are parallel to b *

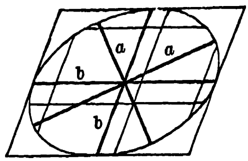


Fig 192

Any two conjugate diameters form with the straight line at infinity a self-conjugate triangle (Art 258), of which one vertex is the centre of the conic and the other two are at infinity

Since in a self-conjugate triangle two of the sides cut the conic and the third side does not (Art 262), and since the straight line at infinity cuts a hyperbola but does not cut an ellipse, it follows that of every two conjugate diameters of a hyperbola one only cuts the curve, while an ellipse is cut by all its diameters

1/ 285 PROBLEM *Given five points A, B, C, D, E on a conic, to determine its centre*

Solution We have only to repeat the construction given in Art 257, II (right), assuming the straight line s to lie in this case at infinity Draw through C a parallel to AB, and determine the point C' in which this parallel meets the conic again, draw also through B a parallel to AC, and determine the point B' in which this parallel meets the conic again The straight line u which joins the points of intersection of the pairs of opposite sides of the quadrangle $ACBC'$, and the straight line v which joins the points of intersection of the pairs of opposite sides of the quadrangle $ABCB'$, will meet in the required point O, which is the pole of the straight line at infinity and therefore the centre of the conic †

The straight lines u and v are the diameters conjugate respectively to AB and AC, if through O there be drawn the straight lines u', v' parallel to AB, AC respectively, then u and u', v and v' will be two pairs of conjugate diameters

If the conic is determined by five tangents, its centre may be found by a method which will be explained further on (Art 319)

* APOLLONIUS, *loc cit*, lib II 20

† If u and v should be parallel, the conic is a parabola, whose diameters are parallel to u and v

286 Four tangents to a conic form a complete quadrilateral whose diagonals are the sides of a self-conjugate triangle (Art 260) Suppose the four tangents to be parallel in pairs (Fig 191), then one diagonal will pass to infinity, and consequently the other two will be conjugate diameters (Art 284), thus

The diagonals of any parallelogram circumscribed to a conic are conjugate diameters

The points of contact of the four tangents form a complete quadrangle whose diagonal points are the vertices of the self-conjugate triangle (Arts 169, 260) In the case where the four tangents are parallel in pairs one of these diagonal points is the centre of the conic, and the other two lie at infinity That is to say, the six sides of the quadrangle are the sides and diagonals of an inscribed parallelogram, its sides are parallel in pairs to the diagonals of the circumscribed parallelogram, and its diagonals intersect in the centre of the conic

287 Conversely, let $ABA'B'$ (Fig 191) be any inscribed parallelogram, and consider it as a complete quadrangle Since its three diagonal points must be the vertices of a self-conjugate triangle, one of them will be the centre of the conic, and the other two will be the points at infinity on two conjugate diameters, thus

In any parallelogram inscribed in a conic, the sides are parallel to two conjugate diameters and the diagonals intersect in the centre

Or again

The chords which join a variable point A on a conic to the extremities B and B' of a fixed diameter are always parallel to two conjugate diameters

288 The following conclusions can be drawn at once from Art 286

Any two parallel tangents (a and a') are cut by any pair of conjugate diameters in two pairs of points, the straight lines connecting which give two other parallel tangents (b and b')

If from the extremities (A and A') of any diameter straight lines be drawn parallel to any two conjugate diameters, they will meet in two points on the curve, and the chord joining these will be a diameter

Given any two parallel tangents a and a' whose points of

contact are A and A' respectively, and any third tangent b, if from A a parallel be drawn to the diameter passing through $a'b$ this parallel will meet the tangent b at its point of contact B

Given any two parallel tangents a and a' whose points of contact are A and A' respectively, and another point B on the conic, the tangent at B will meet the tangent a in a point lying on that diameter which is parallel to $A'B$, and it will meet the tangent a' in a point lying on that diameter which is parallel to AB

289 Suppose now that the conic is a circle (Fig 193), i e the locus of the vertex of a right angle AMB whose arms AM and BM turn round fixed points A and B respectively These arms in moving generate two equal and consequently projective pencils, therefore the tangent at A will be the ray of the first pencil which corresponds to the ray BA of the second (Art 143) The tangent at A must therefore make a right angle with BA, and similarly the tangent at B will be perpendicular to AB The tangents at A and B are therefore parallel, and consequently AB is a diameter, and the middle point O of AB is the centre of the circle (Art 282)

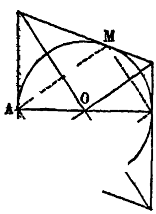


Fig 193

I Since AB is a diameter, the straight lines AM and BM will be parallel to a pair of conjugate diameters, whatever be the position of M (Art 287), therefore

Every pair of conjugate diameters of a circle are at right angles to one another

II Since the diagonals of any parallelogram circumscribed about the circle are conjugate diameters, they will intersect at right angles, thus *any parallelogram which circumscribes a circle must be a rhombus*

III In a rhombus, the distance between one pair of opposite sides is equal to the distance between the other pair, thus by allowing one pair of opposite sides of the circumscribed rhombus to vary while the other pair remain fixed, we see that the distance between two parallel tangents is constant This distance is the length of the straight line joining the points of contact of the tangents, for this straight line, which is a

diameter, cuts at right angles the conjugate diameter and the tangents parallel to it, therefore *all diameters of a circle are equal in length*

IV The diagonals of any inscribed parallelogram are diameters, but all diameters are equal in length, therefore *any parallelogram inscribed in a circle must be a rectangle*

290 Returning to the general case where the conic is any whatever (Fig 189), let s be any straight line and S its pole All chords parallel to s will be bisected by the diameter passing through S, for since S and the point at infinity on s are conjugate points with respect to the conic, the polar of the second point will pass through the first We may also say that

If a diameter pass through a fixed point, the conjugate diameter will be parallel to the polar of this point

I If the diameter passing through S cuts the conic in two points M and M', then MM' is divided harmonically by the pole S and the polar s*, thus if O is the middle point of MM', that is, the centre of the conic, and R the point where MM' is cut by the polar s, we have (Art 69)

$$OS \cdot OR = OM^2$$

II From this follows a *construction for the semi-diameter conjugate to a chord AA' of a conic, having given the extremities A and A' of the chord and three other points on the conic* We determine (Art 285) the centre O, and join it to the middle point R of AA', we then construct the tangent at A and take its point of intersection S with OR If now a point M be taken on OR such that OM is the mean proportional between OR and OS, then OM will be the required semi-diameter

If O lie between R and S, so that OR and OS have opposite signs, the diameter OR will not cut the conic, but in this case also the length OM, the mean proportional between OR and OS, is called the *magnitude of the semi-diameter* conjugate to the chord AA'

An analogous definition can be given for the case of any straight line (Art 294)

III If the conic is a circle, the perpendicularity of the conjugate diameters in this case gives the theorem

* APOLLONIUS, *loc cit*, i 34, 36, ii 29, 30

The polar of any point with respect to a circle is perpendicular to the diameter which passes through the pole

291 From this last property can be derived a second demonstration of the very important theorem of Art 263 (3), *viz*

The range formed by any number of collinear points, and the pencil formed by their polars with respect to any given conic, are two projective forms

Consider as poles the points $A, B, C, \ldots$ lying on a straight line s (Fig. 194), the diameters $O\,(A, B, C, \ \ )$ obtained by joining them to the centre O of the conic will form a pencil which is in perspective with the range $A, B, C,$ Another pencil will be formed by the polars $a, b, c,$ of the points $A, B, C,$ since these polars all pass through a point S (Art 256), the pole of s If now the conic is a circle,

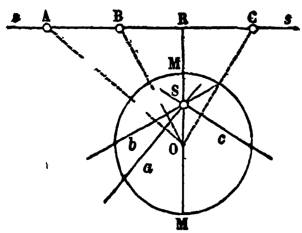


Fig 194

by the property proved in Art 290, III, the straight lines $(A, B, C, \ \ )$ are perpendicular respectively to $a, b, c,$, and the two pencils are in this case equal The range of poles $A, B, C,$ is therefore projective with the pencil of polars $a, b, c,$ with regard to a circle

This result may now be extended and shown to hold not only for a circle but for any conic For any given conic may be regarded as the projection of a circle (Arts 149, 150) In the projection, to harmonic forms correspond harmonic forms (Art 51), consequently to a point and its polar with regard to the conic will correspond a point and its polar with regard to the circle, and to a range of poles and the pencil formed by their polars with regard to the conic will correspond a range of poles and the pencil formed by their polars with regard to the circle But it has been seen that this range and pencil are projective in the case of the circle, therefore the same is true with regard to the range and pencil in the case of the conic, and the theorem is proved

292 THEOREM *A quadrangle is inscribed in a conic, and a point is taken on the straight line which joins the points of intersection of the pairs of opposite sides If from this point be drawn the straight lines connecting it with the two pairs of opposite vertices, and also a pair of*

tangents to the conic, these straight lines will be three conjugate pairs of an involution

Let $ABCD$ be a simple quadrangle inscribed in a conic (Fig 195), let the diagonals AC, BD meet in F, and the pairs of opposite sides BC, AD and AB, CD in E and G respectively, the points E, F, G will then be conjugate two and two with respect to the conic (Art 259) Take any point I on EG and join it to the vertices of the quadrangle, and draw also the tangents IP, IQ to the conic The two tangents are harmonically separated by IE, IF (Art 264), since these are conjugate straight lines, F being the pole of IE But the rays IE, IF are harmonically conjugate also with regard to IA, IC, for the diagonal AC of the complete quadrilateral formed by AB, BC, CD, and DA is divided harmonically by the other two diagonals BD and EG, and the two pairs of rays in question are formed by joining I to the four harmonic points on AC For a similar reason the rays IE, IF are harmonically conjugate with regard to IB, ID The pair of tangents, the rays IA, IC, and the rays IB, ID are therefore three conjugate pairs of an involution, of which IE, IF are the double rays (Art 125)

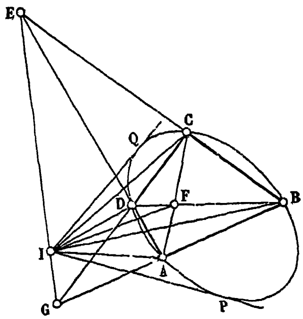


Fig 195

I By virtue of the theorem correlative to that of Desargues (Art 183, right), a conic can be inscribed in the quadrilateral $ABCD$ so as to touch the straight lines IP and IQ

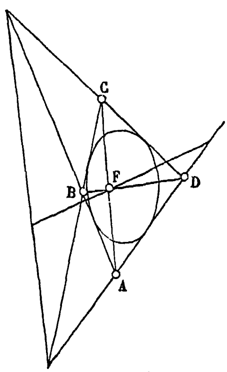


Fig 196

II The theorem correlative to the one proved above may be thus enunciated

If a simple quadrilateral $ABCD$ (Fig 196) is circumscribed about a conic, and if through the point of intersection of its diagonals any transversal be drawn, this will cut the conic and the pairs of opposite sides AB and CD, BC and AD, in three pairs of conjugate points of an involution

III By virtue of Desargues' theorem (Art 183, left), a conic can be described to pass through the four vertices of the quadrilateral and through the two points where the conic is cut by the transversal *

* CHASLES, *Sections coniques*, Arts 122, 126

293. The theory of conjugate points with regard to a conic gives a solution of the problem

To construct the points of intersection of a given straight line s with a conic which is determined by five points or by five tangents

Take on s any two points U and V, construct their polars u and v (Art. 257), and let U' and V' be the points where these meet s. If the involution determined by the two pairs of reciprocal points U and U', V and V', has two double points M and N, these will be the required points of intersection of the conic with s. If U' and V' should coincide, the conic touches s at the point in which they coincide. If the involution has no double points, the conic does not cut s.*

By a correlative method may be solved the problem *to draw from a given point S a pair of tangents to a conic which is determined by five tangents or by five points*

294. Let A and A' be a pair of points lying on a straight line s which are conjugate with respect to the conic, and let O be the point where s meets the diameter passing through its pole S (the diameter bisecting chords parallel to s). Then O will be the centre of the involution formed on s by the pairs of conjugate points such as A , 125)

$$OA \cdot OA' = \text{constant}$$

If s cuts the conic in two points M and N, these will be the double points of the involution, and

$$OA \cdot OA' = OM^2 = ON^2$$

If s does not cut the conic, the constant value of $OA \cdot OA'$ will be negative (Art 125), in this case there exists a pair H and H' of conjugate points of the involution, or of conjugate points with regard to the conic, such that O lies midway between them and

$$OA \cdot OA' = OH \cdot OH' = -OH^2 = \quad OH'$$

The segment HH' has been called an *ideal chord*† of the conic, just as MN in the first case is a real chord. Accepting this definition we may say that a diameter contains the middle points of all chords, real and ideal, which are parallel to the conjugate diameter.

When two conics are said to have a real common chord MN it is meant that they both pass through the points M and N. When two conics are said to have an ideal common chord HH' this signifies that H and H' are conjugate points with regard to both conics, and that the diameters of the two conics which pass through the respective poles of HH' both pass through the middle point of HH'.

* STAUDT *Geometrie der Lage*, Art 305

† PONCELET, *loc cit* p 29

295 A pencil of rays in involution has in general (Art 207) one pair of conjugate rays which include a right angle. Therefore

Through a given point can always be drawn one pair of straight lines which are conjugate with respect to a given conic and which include a right angle, and these are the internal and external bisectors of the angle made with one another by the tangents drawn from the given point, when this is exterior to the conic

296 In Art 263 (Fig 183) let the point G be taken to coincide with the centre O of the conic (hyperbola or ellipse), two conjugate lines such as GF, GE will then become conjugate diameters, and we see that *the pairs of conjugate diameters of a conic form an involution* If the conic is a hyperbola, the asymptotes are the double rays of the involution (Arts 264, 283), thus *any two conjugate diameters of a hyperbola are harmonically conjugate with regard to the asymptotes* * If the conic is an ellipse, the involution has no double rays

Consider two pairs of conjugate elements of an involution, the one pair either overlaps or does not overlap the other, and according as the first or the second is the case, the involution has not, or it has, double points (Art 128), thus

Of any two pairs of conjugate diameters of an ellipse, the one aa' is always separated by the other bb' (Fig 192),

Of any two pairs of conjugate diameters of a hyperbola, the one aa' is never separated by the other bb' (Fig 197)

297 The involution of conjugate diameters will have one pair of conjugate diameters including a right angle (Art 295) If there were a second such pair, every diameter would be perpendicular to its conjugate (Art 207), and in that case the angle subtended at any point on the curve by a fixed diameter would be a right angle (Art 287), and consequently the conic would be a circle Every conic therefore which is not a parabola or a circle has a single pair of conjugate diameters which are at right angles to one another These two diameters a and a' are called the *axes* of the conic (Figs 192, 197) In the

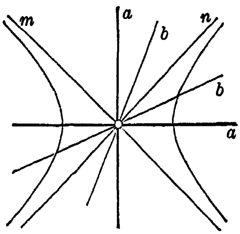


Fig 197

* DE LA HIRE, *loc cit*, book ii prop 13, Cor 4

hyperbola (Fig 197) the axes are the bisectors of the angle between the asymptotes m and n (Arts 296, 60)

In the ellipse both axes cut the curve (Art 284), the greater (a') is called the *major*, the smaller (a) the *minor* axis In the hyperbola only one of the axes cuts the curve, this one (a') is called the *transverse* axis, the other (a) the *conjugate* axis The points in which the conic is cut by the axis a' in either case are called the *vertices*

Regarding an axis as a diameter which bisects all chords perpendicular to itself, it is seen that the parabola also has an axis For since all chords at right angles to the common direction of the diameters are parallel to one another, their middle points lie on one straight line, which is the axis a of the parabola (Fig 190) The parabola has one vertex at infinity, the other, the finite point in which the axis a cuts the curve, is generally called *the* vertex of the parabola

298 Since each of the orthogonal conjugate diameters of a ...hyperbola) bisects all chords perpen- ...ows that the conic is symmetrical with re- ... the diameters in question (Art 76) The ellipse ... hyperbola have therefore each two axes of symmetry, the parabola, on the other hand, has only one such axis

The ellipse and hyperbola are also symmetrical with respect to a point, the centre of symmetry being in each case the pole of the straight line at infinity

In general, given a conic, a point S, and s the polar of S with respect to the conic, if S be taken as centre and s as axis of harmonic homology (Art 76), the conic is homological with itself (Art 250)*

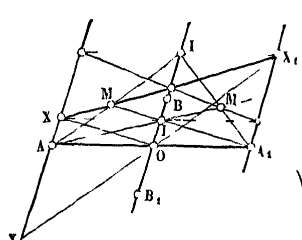


Fig 198

299 In the theorem of Art 275 suppose the inscribed triangle to be AA_1M (Fig 198), that is, let two of its vertices A and A_1 be collinear with the centre O of the conic which is taken to be an ellipse or hyperbola The pole of the side AA_1 will be the point at infinity common to the chords bisected by the diameter AA_1, and the theorem will become the following

* See also Art 396, below

The straight lines which join two conjugate points P and P' to the extremities A and A_1 of that diameter whose conjugate is parallel to PP' intersect on the conic

300 The pairs of conjugate points taken, similarly to P and P', on the diameter conjugate to AA_1 form an involution (Art 263) whose centre is the centre O of the conic If this involution has two double points B and B_1, these lie on the curve, which is therefore an ellipse If the involution has no double points, the conic is a hyperbola (Art 284), in this case two points B and B_1 can be found which are conjugate in the involution and consequently conjugate with respect to the conic, and which lie at equal distances on opposite sides of O (Art 125) In both cases the *length* of the diameter conjugate to AA_1 is interpreted as being the segment BB_1 (Arts 290, 294)

In the ellipse we have (Art 294)

$$OP \quad OP' = \text{constant} = OB^2 = OB_1^2,$$

and in the hyperbola

$$OP \ OP' = \text{constant} = OB \ OB_1 = -OB^2 = -OB_1^2$$

301 The foregoing theorem enables us to solve the problem

To construct by points a conic, having given a pair of conjugate diameters AA_1 and BB_1 in magnitude and position

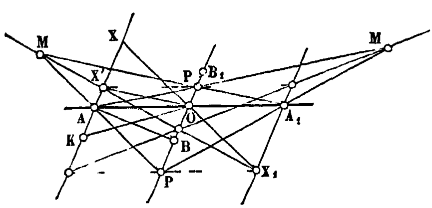


Fig 199

In the case of the ellipse (Fig 198) the four points A, A_1, B, B_1 all lie on the curve, in the case of the hyperbola (Fig 199) let AA_1 be that one of the two given diameters which meets the conic

Construct on the diameter BB_1 several pairs of conjugate points P and P' of the involution determined by having O as centre and B and B_1 in the first case as double points, in the second case as conjugate points The straight lines AP and A_1P' (as also A_1P and AP') will intersect on the curve

302 The straight lines OX, OX' drawn parallel to AP, A_1P' respectively are a pair of conjugate diameters (Art 287) The

pairs of conjugate diameters form an involution (Art. 296), consequently the pairs of points analogous to X, X' (in which the diameters cut the tangent at A) also form an involution, the centre of which is A, since OA and the diameter OB parallel to AX are a pair of conjugate diameters If the conic is a hyperbola, the involution of conjugate diameters has two double rays, which are the asymptotes, therefore the points K and K_1, in which AX meets the asymptotes, are the double points of the involution XX', *

303 Since $OPAX$ is a parallelogram, $AX = -OP$, and from the similar and equal triangles $OP'A_1$ and $AX'O$, $AX' = OP'$ † But $OP\ OP' = \pm OB^2$ (Art 125), therefore $AX\ AX' = \mp OB^2$, or

The rectangle contained by the segments intercepted on a fixed tangent to a conic between its point of contact and the points where it is cut by any two conjugate diameters is equal to the square ($\mp OB^2$) on the semi-diameter drawn parallel to the tangent

304 We have seen (Art 302) that in the case of the hyper- he double points of the involution of which X, X' a pair of conjugate points, thus

$$AX\ AX' = AK^2 = OB^2$$

Therefore $AK = OB$, and $OAKB$ is a parallelogram Accordingly

If a parallelogram be described so as to have a pair of conjugate semi-diameters of a hyperbola as adjacent sides, one of its diagonals will coincide with an asymptote ‡

Further, *the other diagonal AB is parallel to the second asymptote* For consider the harmonic pencil (Art 296) formed by the two asymptotes and the two conjugate diameters OA, OB The four points in which this pencil cuts AB will be harmonic, but one of the asymptotes OK meets AB in its middle point, therefore the other will meet it at infinity (Art 59)

305 Let X_1 be the point where the diameter OX meets the tangent at A_1 Since OX' and OX_1 are a pair of conjugate lines which meet in a point on the chord of contact AA_1 of

* In Fig 199 only one of the points K, K_1 is shown

† In order to account for the signs, it need only be observed that in the case of the ellipse OP and OP' are similar, but AX and AX' opposite to one another in direction, while in the case of the hyperbola OP and OP' are opposite, but AX and AX' similar as regards direction

‡ APOLLONIUS, *loc cit*, book II I

the tangents AX and A_1X_1, the straight line $X'X_1$ (Art. 274) will be a tangent to the conic

The point of contact of this tangent is M, the point of intersection of AP and A_1P' (Art 299)

306 It is seen moreover that $X'X_1$ is one diagonal of the parallelogram formed by the tangents at A and A_1 and the parallels to AA_1 drawn through P and P', this may also be proved in the following manner All points of a diameter have for their polars straight lines which are parallel to the conjugate diameter (Art. 284), if then through the conjugate points P and P' parallels be drawn to AA_1, the first will be the polar of P' and the second the polar of P, consequently these parallels are conjugate lines If now the theorem of Art 274 be applied to these conjugate lines and the two tangents at A and A_1, we obtain the following proposition

If a parallelogram is such that one pair of its opposite sides are tangents to a conic, and the other pair are straight lines, conjugate with regard to the conic and drawn parallel to the chord of contact of the two tangents, then its diagonals also will be tangents to the conic

307 This gives the following solution of the problem

To construct a conic by tangents, having given a pair of conjugate diameters AA_1 and BB_1 in magnitude and direction

Suppose BB_1 to be that diameter which meets the conic in the case where the latter is a hyperbola On BB_1 determine a pair of conjugate points P and P' of the involution which has the centre O of the conic as centre and the points B, B_1 either as double points or as conjugate points, according as the conic to be drawn is an ellipse or a hyperbola Draw through A and A_1 parallels to BB_1, and through P and P' parallels to AA_1, the diagonals of the parallelogram so obtained will be tangents to the required conic

308 The segments AX and A_1X_1 are equal in magnitude and opposite in sign, and it has been seen that $AX \cdot AX' = \mp OB^2$, therefore $AX' \cdot A_1X_1 = \pm OB^2$, or

*The rectangle contained by the segments intercepted upon two parallel fixed tangents between their points of contact and the points where they are cut by a variable tangent ($X'X_1$) is equal to the square ($\pm OB^2$) on the semi-diameter parallel to the fixed tangents**

309 Since the straight line OB is parallel to AX and A_1X_1 and half-way between them, the segments determined by AM

* See Art 160

and A_1M respectively on A_1X_1 and AX (measured from A_1 and A respectively) are double of OP and OP', but by the theorem of Art 300 the rectangle $OP\ OP'$ is constant, thus

The straight lines connecting the extremities of a given diameter with any point on the conic meet the tangents at these extremities in two points such that the rectangle contained by the segments of the tangents intercepted between these points and the points of contact is constant *

310 Since X is (Art 288) the point of intersection of the tangent at A and the tangent parallel to $X'X_1$, the proposition of Art 303 may also be expressed as follows

The rectangle contained by the segments (AX, AX') determined by two variable parallel tangents upon any fixed tangent is equal to the square ($\mp OB^2$) on the semi-diameter parallel to the fixed tangent

311 From the theorems of Arts 299, 300 is derived the solution of the following problem

Given the two extremities A and A_1 of a diameter of a conic, a third point M on the conic, and the direction of the diameter conjugate to AA_1, to determine the length of the latter diameter (Fig 199)

Through O, the middle point of AA_1, draw the diameter whose direction is given, let it be cut by AM and A_1M in P and P' respectively, and take OB the mean proportional between OP and OP', then OB will be the half of the length required

312 The proposition of Art 303 gives a *construction for pairs of conjugate diameters, and in particular for the axes, of an ellipse of which two conjugate semi diameters OA and OB are given in magnitude and direction* (Fig 200)

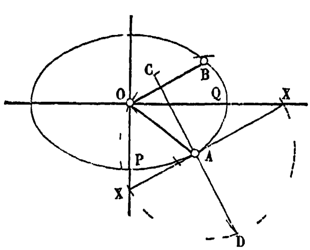


Fig 200

Through A draw a parallel to OB, this will be the tangent at A and will be cut by any two conjugate diameters in two points X and X' such that

$$AX\ AX' = -OB^2$$

If now there be taken on the normal at A two segments AC and AD each equal to OB, every circle passing through C and D will cut this tangent in two points X and X' which possess the property expressed by the above equation, these points are therefore such that the straight lines joining them to the centre O will give the directions of a pair of conjugate diameters If the circle be drawn

* APOLLONIUS, *loc cit*, lib III 53

through O the angle XOX' becomes a right angle, and consequently OX, OX' will be the directions of the axes

Since the circular arcs CX', $X'D$ are equal, the angles COX', $X'OD$ are equal, consequently OX', OX are the internal and external bisectors of the angle which OC, OD make with one another In order then to construct the semi-axes OP, OQ in magnitude, let fall perpendiculars AX_1, AX_1' on OX, OX' respectively Then X and X_1, X' and X_1' are pairs of conjugate points, therefore OP will be the geometric mean between OX and OX_1, and OQ the geometric mean between OX' and OX_1' *

313 Through the extremities A and A' (Fig 201) of two conjugate semi-diameters OA and OA' of a conic draw any two parallel chords AB and $A'B'$ To find the points B and B' we have only to join the poles of these chords, this will give the diameter OX' which passes through their middle points

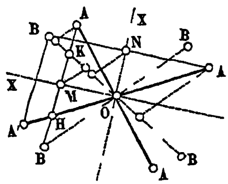


Fig 201

Let OX be the diameter conjugate to OX', i e that diameter which is parallel to the chords AB, $A'B'$ The pencils $O(XX'AB)$ and $O(X'XA'B')$ are each harmonic (Art 59), and are therefore projective with one another, consequently the pairs of rays $O(XX', AA', BB')$ are in involution (Art 123) But the two pairs $O(XX', AA')$ determine the involution of conjugate diameters (Arts 127, 296), therefore also OB and OB' are conjugate diameters Thus

If through the extremities A and A' of two conjugate semi-diameters parallel chords AB, $A'B'$ be drawn, the points B and B' will be the extremities of two other conjugate semi-diameters

Two diameters AA and BB determine four chords AB which form a parallelogram (Arts 260, 287) The diameters conjugate respectively to them form in the same way another parallelogram, which has its sides parallel to those of the first, that is every chord AB is parallel to two chords $A'B'$, and not parallel to two other chords $A'B'$

314 Let H, K be the points where AB is cut by OA', OB' respectively The diameter OX' which bisects $A'B'$ will also bisect HK, therefore AB and HK have the same middle point, thus $AH = KB$ and $AK = HB$ The triangles OAK and OBH

* CHASLES *Aperçu historique*, pp 45, 362, *Sections coniques*, Art 205

are therefore equal in area (Euc I 37), as also AKB' and BHA', and therefore also OAB' and $OA'B$ are equal Accordingly

The parallelogram described on two semi-diameters (OA, OB') as adjacent sides is equal in area to the parallelogram described similarly on the two conjugate semi-diameters

In the same way the triangles OAB and $OA'B'$ can be proved equal

The triangles AHA', BKB' are equal for the same reason, and OAH, OBK are equal, and therefore also OAA' and OBB' Therefore

*The parallelogram described on a pair of conjugate semi-diameters as adjacent sides is of constant area**

315 Let M and N be the middle points of the non-parallel chords AB and $A'B'$ Since AB and $A'B'$ are parallel to a pair of conjugate diameters (Art 287) and since ON is the diameter conjugate to the chord $A'B'$, therefore ON will be parallel to AB, so also OM will be parallel to $A'B'$ The angles OMA and ONA' are therefore equal or supplementary, and since the triangles OMA and ONA' are equal in area (being halves of the equal triangles OAB and $OA'B'$), we have (Euc VI 15),

$$OM \cdot AM = \pm ON \cdot NA' \dagger$$

Now project (Fig 202) the points A, M, B, A', N, B' from the point at infinity on OB as centre upon the straight line $B'B'$ The ratio of the parallel segments AM and ON, OM and NA' is equal to that of their projections, we conclude therefore from the equality just proved that the [illegible] contained by the projections of OM and AM is equal to that contained by the projections of ON and NA' As the projecting rays are parallel to OB, the projections of OM and MA are

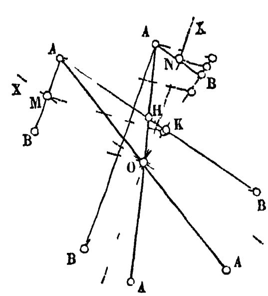


Fig 202

* APOLLONIUS, *loc cit*, lib VII 31, 32

† The signs + and − caused by the relative direction of the segments OM, NA and ON, AM correspond respectively to the case of the ellipse (Fig 201) and to that of the hyperbola (Fig 202)

each equal to half the projection of BA or of OA. Since N is the middle point of $A'B'$, the projection of ON will be equal to half the sum of the projections of OA' and OB', and the projection of NA' will be equal to half the projection of $B'A'$, that is, to half the difference between the projections of OA' and OB' We have therefore

$$(\text{proj } OA)^2 = \pm \text{proj } (OA' + OB') \times \text{proj } (OB' - OA'),$$

or

$$(\text{proj } OA')^2 \pm (\text{proj } OA)^2 = (\text{proj } OB')^2$$

In the same manner, by projecting the same points on OB by means of rays parallel to OB' (Fig 203), we should obtain

$$(\text{proj } OA)^2 \pm (\text{proj } OA')^2 = (\text{proj } OB)^2$$

This proves the following proposition

If any pair of conjugate diameters are projected upon a fixed diameter by means of parallels to the diameter conjugate to this last, then the sum (in the ellipse) or difference (in the hyperbola) of the squares on the p[illegible] is equal to the square on the fixed diameter

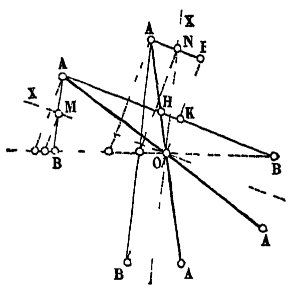


Fig 203

By the Pythagorean theorem (Euc I 47) the sum of the squares on the orthogonal projections of a segment on two straight lines at right angles to one another is equal to the square on the segment itself If then a pair of conjugate diameters are projected orthogonally on one of the axes of a conic and the squares on the projections of each diameter on the two axes are added together, the following proposition will be obtained

*The sum (for the ellipse) or difference (for the hyperbola) of the squares on any pair of conjugate diameters is constant, and is equal to the sum or the difference of the squares on the axes**

316 If five points on a conic are given, then by the method explained in Art 285 the centre O and two pairs of conjugate diameters u and u', v and v' can be constructed If these pairs overlap one another, the conic is an ellipse, in the contrary case it

* APOLLONIUS *loc cit*, lib VII 12, 13, 22, 25

is a hyperbola (Art 296) If in this second case the double rays of the involution determined by the two pairs u and u', v and v' be constructed, they will be the asymptotes of the hyperbola

If in either case the orthogonal pair of conjugate rays of the involution be constructed, they will be the axes of the conic

The direction of the axes can be found without first constructing the centre and two pairs of conjugate diameters * Let A, B, C, F, G be the five given points (Fig 168), describe a circle round three of them ABC, and construct (Art 227, I) the fourth point of intersection C' of this circle with the conic determined by the five given points Any transversal will cut the two curves and the two pairs of opposite sides of the common inscribed quadrangle $ABCC'$ in pairs of points forming an involution (Art 183) The double points P and Q (if such exist) of this involution will be conjugate with regard to each of the curves (Arts 125, 263), i e they will be the pair common (Art 208) to the two involutions which are formed on the transversal by the pairs of points conjugate with regard to the circle and by the pairs of points conjugate with regard to the conic (Art 263) Suppose that the straight line at infinity is taken as the transversal As this straight line does not meet the circle, one at least of these two involutions will have no double points, and consequently (Art 208) the points P and Q do really exist Since these points are infinitely distant and are conjugate with regard to both curves they will be (Arts 276, 284) the poles of two conjugate diameters of the circle and also of two conjugate diameters of the conic, but conjugate diameters of the circle are perpendicular to one another (Art 289), therefore P and Q are the poles of the axes of the conic Further, the segment PQ is harmonically divided by either pair of opposite sides of the quadrangle $ABCC'$, consequently P and Q are the points at infinity on the bisectors of the angles included by the pairs of opposite sides (Art 60) In order then to find the required directions of the axes, we have only to draw the bisectors † of the angle included by a pair of opposite sides of the quadrangle $ABCC'$, for example by AB and CC' (Fig 168)

317 Let $qrst$ (Fig 161) be a complete quadrilateral and S any point It has already been seen (Art 185 right) that the pairs of rays a and a', b and b', which join S to two pairs of opposite vertices, belong to an involution of which the tangents drawn from S to any conic inscribed in the quadrilateral are a pair of conjugate rays Suppose the involution to have two double rays m and n, they will be harmonically conjugate

* PONCELET, *loc cit*, Art 394.

† See also the note to Art 387

with regard to such a pair of tangents (Art. 125), and will consequently be conjugate lines with respect to the conic But (Art 218, right) m and n are the tangents at S to the two conics which can be inscribed in the quadrilateral $qrst$ so as to pass through S Therefore

If two conics which are inscribed in a given quadrilateral pass through a given point, their tangents at this point are conjugate lines with respect to any conic inscribed in the quadrilateral

Instead of taking an arbitrary point S, let m be supposed given If this straight line does not pass through any of the vertices of the quadrilateral, there will be one conic, and only one, which touches the five straight lines m, q, r, s, t (Art 152) Let S be the point where this conic touches m, there will be a second conic which is inscribed in the quadrilateral and which passes through S, let the tangent to this at S be n The straight lines m and n will then be conjugate to one another with respect to all conics inscribed in the quadrilateral, and therefore (Art 255),

The poles of any straight line m with respect to all conics inscr in the same quadrilateral lie on another straight line n

Moreover, since the straight lines m and n are the double rays of the involution of which the rays drawn from S to two opposite vertices are a conjugate pair, therefore *m and n divide harmonically each diagonal of the quadrilateral*

318 I The correlative propositions to those of Art 317 are the following

If a straight line touches two conics which circumscribe the same quadrangle, the two points of contact are conjugate to one another with respect to all conics circumscribing the quadrangle

The polars of any given point M with respect to all the conics circumscribing the same quadrangle meet in a fixed point N The segment MN is divided harmonically at the two points where it is cut by any pair of opposite sides of the complete quadrangle

II Suppose in the second theorem of Art 317 that the straight line m lies at infinity, then the poles of m will be the centres of the conics (Art 281), and n will bisect each of the diagonals of the quadrilateral (Art 59), therefore

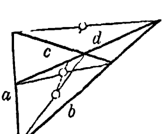


Fig 204

The centres of all conics inscribed in the same quadrilateral lie

on the straight line (Fig 204) *which passes through the middle points of the diagonals of the quadrilateral* *

III. Suppose similarly in theorem I of the present Article that the point M lies at infinity, the polars of M will become the diameters conjugate to those which have M as their common point at infinity, thus

In any conic circumscribing a given quadrangle, the diameter which is conjugate to one drawn in a given fixed direction will pass through a fixed point

319 Newton's theorem (Art. 318, II) gives a simple *method for finding the centre of a conic determined by five tangents* a, b, c, d, e (Fig 205) The four tangents a, b, c, d form a quadrilateral, join the middle points of its diagonals Let the same be done with regard to the quadrilateral $abce$, the two straight lines thus obtained will meet in the required centre O

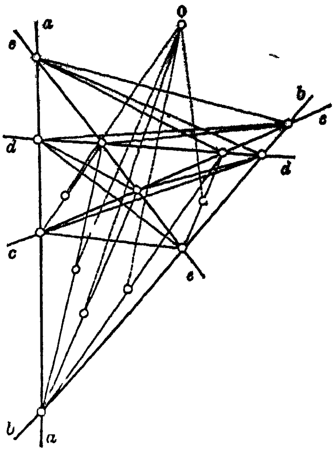


Fig 205

The five tangents, taken four and four together, form five quadrilaterals, the five straight lines which join the middle points of the diagonals of each of the quadrilaterals will therefore all meet in the centre O of the conic inscribed in the pentagon $abcde$

The same theorem enables us *to find the direction of the diameters of a parabola which is determined by four tangents* a b c d For each point on the straight line joining the middle points of the diagonals of the quadrilateral $abcd$ is the pole of the straight line at infinity with regard to some conic inscribed in the quadrilateral (Art 318 II) therefore the point at infinity on the line will be the pole with regard to the inscribed parabola (Arts 2[illegible]1 III and 2[illegible]3) The straight line therefore which joins the middle points of the diagonals is itself a diameter of the parabola (Fig 204)

* Newton, *Principia* book I lemma 25 Cor 3

CHAPTER XXII.

POLAR RECIPROCAL FIGURES

320 AN auxiliary conic K being given, it has been seen (Art 256) that if a variable pole describes a fixed straight line its polar turns round a fixed point, and reciprocally, that if a straight line considered as polar turns round a fixed point, its pole describes a fixed straight line

Consider now as polars all the tangents of a given curve C or in other words suppose the polar to move, and to e... the given curve Its pole will describe another curve, w... may be denoted by C′ Thus the points of C′ are the pole... the tangents of C

But it is also true that, reciprocally, the points of C are the poles of the tangents of C′ For let M' and N' be two points on C′ (Fig 206), their polars m and n will be two tangents to C and the point mn where they meet will be the pole of the chord $M'N'$ (Art 256) Now suppose the point N' to approach M' indefinitely, the chord $M'N'$ will approach more and more nearly to the position of the tangent at M' to the curve C′, the straight line n will at the same time approach more and more nearly to coincidence with m, and the point mn will tend more and more to the point where m touches C In the limit, when the distance $M'N'$ becomes indefinitely small, the tangent to C′ at M' will become the polar of the point of contact of m with C Just then as the tangents of C are the polars of the points of C′, so also are the tangents of C′ the polars of the points of C, if a straight line m touches the curve C at M, the pole M' of m

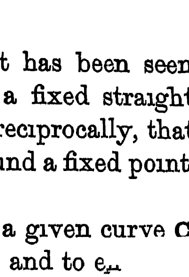


Fig 206

on the straight line (Fig 204) *which passes through the middle points of the diagonals of the quadrilateral* *

III. Suppose similarly in theorem I of the present Article that the point M lies at infinity, the polars of M will become the diameters conjugate to those which have M as their common point at infinity, thus

In any conic circumscribing a given quadrangle, the diameter which is conjugate to one drawn in a given fixed direction will pass through a fixed point

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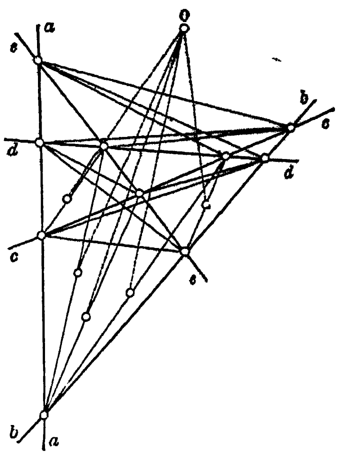

Fig 205

The five tangents, taken four and four together, form five quadrilaterals, the five straight lines which join the middle points of the diagonals of each of the quadrilaterals will therefore all meet in the centre O of the conic inscribed in the pentagon *abcde*

The same theorem enables us *to find the direction of the diameters of a parabola which is determined by four tangents a b c d* For each point on the straight line joining the middle points of the diagonals of the quadrilateral *abcd* is the pole of the straight line at infinity with regard to some conic inscribed in the quadrilateral (Art 318 II), therefore the point at infinity on the line will be the pole with regard to the inscribed parabola (Arts 2[illegible]1 III and 2[illegible]3) The straight line therefore which joins the middle points of the diagonals is itself a diameter of the parabola (Fig 204)

* NEWTON, *Principia* book i lemma 25 Cor 3

CHAPTER XXII.

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320. AN auxiliary conic **K** being given, it has been seen (Art 256) that if a variable pole describes a fixed straight line its polar turns round a fixed point, and reciprocally, that if a straight line considered as polar turns round a fixed point, its pole describes a fixed straight line

Consider now as polars all the tangents of a given curve **C**, or in other words suppose the polar to move, and to envelope the given curve Its pole will describe another curve, which may be denoted by **C′** Thus the points of **C′** are the poles of the tangents of **C**

But it is also true that, reciprocally, the points of **C** are the poles of the tangents of **C′** For let M' and N' be two points on **C′** (Fig 206), their polars m and n will be two tangents to **C** and the point mn where they meet will be the pole of the chord $M'N'$ (Art 256) Now suppose the point N' to approach M' indefinitely, the chord $M'N'$ will approach more and more nearly to the position of the tangent at M' to the curve **C′**, the straight line n will at the same time approach more and more nearly to coincidence with m, and the point mn will tend more and more to the point where m touches **C** In the limit, when the distance $M'N'$ becomes indefinitely small, the tangent to **C′** at M' will become the polar of the point of contact of m with **C** Just then as the tangents of **C** are the polars of the points of **C′**, so also are the tangents of **C′** the polars of the points of **C**, if a straight line m touches the curve **C** at M, the pole M' of m

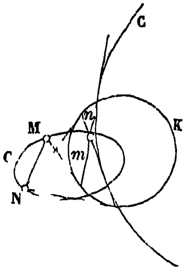

Fig 206

is a point of the curve C′ and the polar m' of M is a tangent to the curve C′ at M'

Two curves C and C′ such that each is the locus of the poles of the tangents of the other, and at the same time also the envelope of the polars of the points of the other, are said to be *polar reciprocals* * one of the other with respect to the *auxiliary conic* K

321. An arbitrary straight line r meets one of the reciprocal curves in n points say, the polars of these points are n tangents to the other curve all passing through the pole R' of r To the second curve therefore can be drawn from any given point R' the same number of tangents as the first curve has points of intersection with the straight line r, the polar of R', and *vice versa* In other words, ***the degree and class of a curve are equal to the class and degree respectively of its polar reciprocal with respect to a conic***

322 Now suppose the curve C to be a conic, and a, b two tangents to it, they will be cut by all the other tangents $c, d, e,$ in corresponding points of two projective ranges (Art 149) In other words, C may be regarded as the curve enveloped by the straight lines $c, d, e,$ which connect the pairs of corresponding points of two projective ranges lying on a and b respectively (Art 150)

The curve C′ will pass through the poles $A', B', C', D', E',$ of the tangents $a, b, c, d, e,$ of C The straight lines $A'(C', D', E',\ )$ will be the polars of the points $a(c, d, e,\ )$ and will form a pencil projective with the range of poles lying on the straight line a (Art 291) so too the straight lines $B'(C', D', E',\ )$ will be the polars of the points $b(c, d, e\ )$ and will form a pencil projective with the range of poles lying on b But the ranges $a(c, d, e\ )$ and $b(c, d, e,\ )$ are projective therefore also the pencils $A'(C', D', E',\ )$ and $B'(C', D', E',\ )$ are projective Consequently C′ is the locus of the points of intersection of corresponding rays of two projective pencils that is (Art 150) a conic Accordingly

The polar reciprocal of a conic with respect to another conic is a conic †

323 When an auxiliary conic K is given and another conic

* PONCELET, *loc cit*, Art 232

† Ibid, Art 231

C whose polar reciprocal C′ is to be determined, the question arises whether C′ is an ellipse, a hyperbola, or a parabola The straight line at infinity is the polar of the centre O of K, therefore the points at infinity on C′ correspond to the tangents of C which pass through O It follows that *the conic* C′ *will be an ellipse or a hyperbola according as the point O is interior or exterior to the conic* C, *and* C′ *will be a parabola when O lies upon* C

If A is the pole of a straight line a with respect to C, and a' the polar of A and A' the pole of a with respect to K, then will A' be the pole of a' with respect to C′, since to four poles forming a harmonic range correspond four polars forming a harmonic pencil (Art 291) and *vice versa* Therefore the centre M' of C′ will be the pole with respect to K of the straight line m which is the polar of O with respect to C To two conjugate diameters of C′ will correspond two points of m which are conjugate with respect to C, &c

324 Let there be given in the plane of the auxiliary conic a *figure* (Art 1) or complex of any kind composed of points, straight lines, and curves, and let the polar of every point, the pole of every line, and the polar reciprocal of every curve, be constructed In this way a new figure will be obtained, the two figures are said to be *polar reciprocals* one of the other, since each of them contains the poles of the straight lines of the other, the polars of its points, and the curves which are the polar reciprocals of its curves To the method whereby the second figure has been derived from the first the name of *polar reciprocation* is given

Two figures which are polar reciprocals one of the other are *correlative figures* in accordance with the law of duality in plane Geometry (Art 33), for to every point of the one corresponds a straight line of the other, and to every range in the one corresponds a pencil in the other They lie moreover in the same plane, their positions in this plane are determinate, but may be interchanged, since every point in the one figure and the corresponding straight line in the other are connected by the relation that they are pole and polar with respect to a fixed conic Thus two polar reciprocal figures are correlative figures which are coplanar, and which have a special relation to one another with respect to their positions in the plane in which they lie On the other hand, if two figures are merely

correlative in accordance with the law of duality, there is no relation of any kind between them as regards their position *

325 If one of the reciprocal figures contains a range (of poles) the other contains a pencil (of polars), and these two corresponding forms are projective (Art 291) If then the points of the range are in involution, the rays of the corresponding pencil will also be in involution, and to the double points of the first involution will correspond the double rays of the second (Art 124) If there is a conic in one of the figures there will also be one in the other figure (Art 322), to the points of the first conic will correspond the tangents of the second, and to the tangents of the first will correspond the points of the second, to an inscribed polygon in the first figure will correspond a circumscribed polygon in the second (Art 320) If the first figure exhibits the proof of a theorem or the solution of a problem, the second will show the proof of the correlative theorem or the solution of the correlative problem, that namely which is obtained by interchanging the elements 'point' and 'line'

326 THEOREM *If two triangles are both self-conjugate with regard to a given conic, their six vertices lie on a conic, and their six sides touch another conic* †

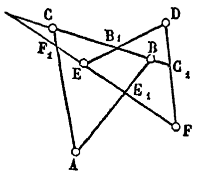

Fig 207

Let ABC and DEF be two triangles (Fig 207) each of which is self conjugate (Art 258) with regard to a given conic **K** Let DE and DF cut BC in B_1 and C_1 respectively, and let AB and AC cut EF in E_1 and F_1 respectively The point B is the pole of CA, and C is the pole of AB, B_1 is the pole of the straight line joining the poles of BC and DE, i e of AF, and C_1 is the pole of the straight line joining the poles of BC and DF, i e of AE The range of poles BCB_1C_1 is therefore (Art 291) projective with the pencil of polars $A(CBFE)$, and therefore with the range of points F_1E_1FE in which this pencil is cut by the transversal EF Thus

$$(BCB_1C_1) = (F_1E_1FE)$$
$$= (E_1F_1EF) \text{ by Art 45,}$$

which shows that the two ranges in which the straight lines BC and EF respectively are cut by AB, CA, DE, FD are projectively related

* STEINER *loc cit*, p vii of the preface, Collected Works vol I p 234

† STEINER, *loc cit*, p 308, § 60, Ex 46, Collected Works, vol I p 448, CHASLES, *Sections coniques*, Art 215

These six straight lines therefore, the six sides of the given triangles, all touch a conic **C** (Art. 150, II).

The poles of these six sides are the six vertices of the triangles, these vertices therefore all lie on another conic **C′** which is the polar reciprocal of **C** with regard to the conic **K** *

327 The proposition of the preceding Article may also be expressed as follows Given two triangles which are self-conjugate with regard to the same conic **K**, if a conic **C** touch five of the six sides it will touch the sixth side also, and if a conic pass through five of the six vertices it will pass through the sixth vertex also

It follows that *if a conic* **C** *touch the sides of a triangle abc which is self-conjugate with regard to another conic* **K**, *there are an infinite number of other triangles which are self-conjugate with regard to the second conic and which circumscribe the first*

For let d be any tangent to **C**, from D, its pole with regard to **K**, draw a tangent e to **C**, and let f be the polar with regard to **K** of the point de, then the triangle def will be self-conjugate with regard to **K** (Art 259) But **C** touches five sides a, b, c, d, e of two triangles which are both self conjugate with respect to **K**, therefore it must also touch the sixth side f, which proves the proposition

328 If the point D is such that from it a pair of tangents e' and f' can be drawn to **K**, the four straight lines e, f, e', f' will form a harmonic pencil (Art 264), since e and f are conjugate straight lines with respect to the conic **K**, consequently the straight lines e' and f' are conjugate to one another with respect to **C**

The locus of D is the conic **C′** which is the polar reciprocal of **C** with regard to **K**, therefore

If a conic **C** *is inscribed in a triangle which is self-conjugate with respect to another conic* **K**, *the locus of a point such that the pairs of tangents drawn from it to the conics* **C** *and* **K** *form a harmonic pencil is a third conic* **C′** *which is the polar reciprocal of* **C** *with respect to* **K**

329 Correlatively *If a conic* **C′** *circumscribes a triangle which is self-conjugate with respect to another conic* **K**, *there are an infinite number of other triangles which are inscribed in* **C′** *and are self conjugate with respect to* **K**, *and the straight lines which cut* **C′** *and* **K** *in two pairs of points which are harmonically conjugate to one another all touch a third conic* **C** *which is the polar reciprocal of* **C′** *with regard to* **K**

* We may show independently that the six vertices lie on a conic as follows. It has been seen that the pencil of polars $A(CBFE)$ is projective with the range of poles BCB_1C_1, it is therefore projective with the pencil $D(BCB_1C_1)$ formed by joining these to the point D Therefore

$$A(CBFE) = D(BCB_1C_1) = D(BCEF)$$
$$= D(CBFE) \text{ by Art 45,}$$

which shows (Art 150, I) that A, B, C, D, E, F lie on a conic

330 THEOREM *If two triangles circumscribe the same conic, their six vertices lie on another conic*

Let $OQ'R'$ and $O'PS$ be two triangles each circumscribing a given conic **C** (Fig 208) The two tangents PS and $Q'R'$ are cut by the four other tangents $O'P, OQ', OR', O'S$ in two groups of corresponding points $PQRS$ and $P'Q'R'S'$ of two projective ranges u and u' (Art 149), consequently the pencils $O(PQRS)$ and $O'(P'Q'R'S')$ formed by connecting these points with O and O' respectively are projective Therefore the points P, Q', R', S, in which their pairs of corresponding rays intersect, lie on a conic **C**$'$ (Art 150, I) passing through the centres O and O', which proves the theorem

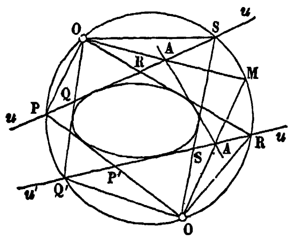

Fig 208

331 The theorem correlative and converse to the foregoing one is the following

If two triangles are inscribed in the same conic, their six sides touch another conic *

This may be proved by considering the triangles $OQ'R'$ and $O'PS$ as both inscribed in the conic **C**$'$, and by reasoning in a manner exactly analogous, but correlative, to that above

332 It follows at once that

If two triangles circumscribe the same conic, the conic which passes through five of their vertices passes through the sixth vertex also	If two triangles are inscribed in the same conic, the conic which touches five of their sides touches the sixth side also

Or

If two conics are such that a triangle can be inscribed in the one so as to circumscribe the other, then there exist an infinite number of other triangles which possess the same property †

333 There are in the figure (Fig 208) four projective forms the two ranges u and u', which determine the tangents to the conic **C**, and the two pencils O and O', which determine the points of **C**$'$, the pencil O is in perspective with the range u

* BRIANCHON *loc cit*, p 35, STEINER, *loc cit*, p 173, § 46, II, Collected Works, vol 1 p 356

† PONCELET, *loc cit*, Art 565

and the pencil O' is in perspective with the range u' If then any tangent to **C** cut the bases u and u' of the two ranges in A and A' respectively, the rays OA and $O'A'$ will meet in a point M lying on **C**$'$, and, conversely, if any point M on **C**$'$ be joined to the centres O and O', the joining lines will cut u and u' respectively in two points A and A' such that the straight line joining them is a tangent to **C** Therefore

If a variable triangle $AA'M$ is such that two of its sides pass respectively through two fixed points O' and O lying on a given conic, and the vertices opposite to them lie respectively on two fixed straight lines u and u', while the third vertex lies always on the given conic, then the third side will touch a fixed conic which touches the straight lines u and u'

If a variable triangle $AA'M$ is such that two of its vertices lie respectively on two fixed tangents u and u' to a given conic, and the sides opposite to them pass respectively through two fixed points O' and O, while the third side always touches the given conic, then the third vertex will lie on a fixed conic which passes through the points O an

334 THEOREM *If the extremities of each of two diagonals of complete quadrilateral are conjugate points with respect to a given conic, the extremities of the third diagonal also will be conjugate points with respect to the same conic* *

Let $ABXY$ (Fig 209) be a complete quadrilateral such that A is conjugate to X, and B to Y, with respect to a given conic **K** (not shown in the figure) Let the sides AB, XY meet in C, and the sides AY, BX in Z, then shall C and Z be conjugate points with respect to the conic **K**

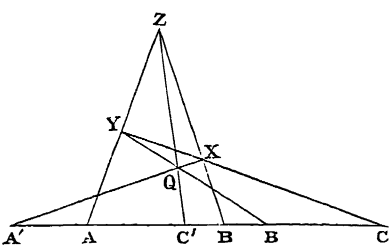

Fig 209

Suppose the polars of the points A, B, C (with respect to **K**) to cut the straight line ABC in A', B', C' respectively The three pairs of conjugate points A and A', B and B', C and C' are in involution, consequently, considering XYZ as a triangle cut by a transversal $A'B'C'$, it follows by Art 135 that the

* HESSE *De octo punctis intersectionis trium superficierum secundi ordinis* (Dissertatio pro venia legendi, Regiomonti, 1840), p 17

straight lines XA', YB', ZC' meet in one point Q Since evidently XA' is the polar of A and YB' the polar of B with respect to K, their point of intersection Q is the pole of AB Since then C is a point on AB and is conjugate to C', its polar will be QC', but QC' passes through Z, therefore C and Z are conjugate points, which was to be proved.

335 The proof of the following, the correlative theorem, is left as an exercise to the student

If two pairs of opposite sides of a complete quadrangle are conjugate lines with respect to a conic, the two remaining sides also are conjugate lines with respect to the same conic

In order to obtain such a complete quadrangle, it is only necessary to take the polar reciprocal of the quadrilateral considered in Hesse's theorem, *i e* the figure which is formed by the polars of the six points A and X, B and Y, C and Z

336 The following proposition is a corollary to that of Art 334

Two triangles which are reciprocal with respect to a conic are in homology *

Let ABC (Fig 210) be any triangle, the polars of its vertices with respect to a given conic form another triangle $A'B'C'$ reciprocal to the first, that is, such that the sides of the first triangle are also the polars of the vertices of the second Let the sides CA and $C'A'$ meet in E, and the sides AB and $A'B'$ in F

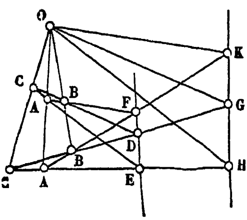

Fig 210

The points B and E are conjugate with respect to the conic, since E lies on $C'A'$, the polar of B, similarly C and F are conjugate points Thus in the quadrilateral formed by BC, CA, AB, and EF, two pairs of opposite vertices B and E, C and F are conjugate, therefore the third pair are conjugate also, viz A and the point D where BC meets EF The polar $B'C'$ of A therefore passes through D, thus BC and $B'C'$ meet in a point D lying on EF Since then the pairs of opposite sides of the two triangles meet one another in three collinear points, the triangles are in homology, and the straight lines AA', BB', CC' which join

* CHASLES, *loc cit*, Art 135

the pairs of vertices meet (Art. 17) in a point O, the pole of the straight line DEF.

337. By combining this theorem with that of Art. 155 the following property may be enunciated:

If two triangles are reciprocals with respect to a given conic **K**, *the six points in which the sides of the one intersect the non-corresponding* sides of the other lie on a conic* **C**, *and the six straight lines which connect the vertices of the one with the non-corresponding vertices of the other touch another conic* **C′**, *the polar reciprocal of* **C** *with respect to* **K** (Art. 322); these straight lines are in fact the polars with regard to **K** of the six points just mentioned.

read

If one of the triangles $A'B'C'$ is inscribed in the other ABC, the three conics **C**, **C′**, and **K** coincide in one which is circumscribed about the former triangle and inscribed in the latter (Arts. 174, 176).

338. PROBLEM. *Given two triangles ABC, $A'B'C'$ which are in homology, to construct (when it exists) the conic with regard t they are reciprocal.*

Take one of the sides, BC for example; the points in which it i cut by $C'A'$ and $A'B'$ are conjugate to the points B and C respectively, and these two pairs of conjugate points determine an involution (Art. 263), the double points of which (if they exist) are the points where BC is cut by the conic in question. In order then to find the points in which this conic cuts BC, it is only necessary to construct these double points. In this way the points in which the sides of the triangles meet the conic can be found, and the latter is determined. Since A' and B are the poles of BC and $C'A'$, these points and that in which $C'A'$ meets BC will be the vertices of a self conjugate triangle (Art. 258). If then, in finding the points of intersection of the conic and the straight lines BC and $C'A'$ in the manner just explained, it should happen that the two involutions found have neither of them double points, the conclusion is that no conic exists such as is required; for if it did exist, it must be cut by two of the sides of the self conjugate triangle (Art. 262).

339. The centre of homology O of the given triangles (Fig. 210) is the pole of the axis of homology DEF, and the projective correspondence (Art. 291) between the points (poles) lying on the axis and the straight lines (polars) radiating from the centre of homology is determined by the three pairs of corresponding elements D and

* Two sides BC and $B'C'$ of the triangles may be termed *corresponding*, when each lies opposite to the pole of the other. And two vertices A and A' may be termed *corresponding*, when each lies opposite to the polar of the other.

AA', E and BB', F and CC' Consequently it is possible to construct with the ruler only (Art 84) the polar of any other point on the axis, and the pole of any other ray passing through the centre O

What has just been said with regard to the point O and the axis of homology may also be said with regard to any vertex of one of the triangles and its polar (the corresponding side of the other triangle) For if *e g* the vertex A' and the side BC be considered, the projective correspondence between the straight lines radiating from A' and the points lying on BC is determined by the three pairs of corresponding elements $A'B'$ and C, $A'C'$ and B, $A'O$ and D

This being premised, it will be seen that the polar of any point P and the pole of any straight line p can be constructed with the help of the ruler only For suppose P to be given, it has been shown that the poles of the straight lines PO, PA, PB, PC, PA', can be constructed, and these all lie on a straight line X which is the required polar of P So again if the straight line p is given, the polars of the points in which it meets BC, CA, can be constructed, and will meet in a point which is the pole of p

It will be noticed that all these determinations of poles and polars are linear (*i e* of the first degree) and independent of the construction (Art 338) of the auxiliary conic, which is of the second degree, since it depends on finding the double elements of an involution The construction of the poles and polars is therefore always possible, even when the auxiliary conic does not exist In other words the two given triangles in homology determine between the points and the straight lines of the plane a reciprocal correspondence such that to every point corresponds a straight line and to every straight line a point, to the rays of a pencil the points of a range projective with the pencil, and *vice versa* Any point and the straight line corresponding to it may be called *pole* and *polar*, and this assemblage of poles and polars, which possesses all the properties of that determined by an auxiliary conic (Art 254), may be called a *polar system*

Two triangles in homology accordingly determine a polar system If an auxiliary conic exists, it is the locus of the points which lie on the polars respectively corresponding to them, and it is at the same time the envelope of the straight lines which pass through the poles respectively corresponding to them If no auxiliary conic exists, there is no point which lies on its own polar *

* STAUDT, *loc cit*, Art 241

CHAPTER XXIII.

FOCI *

340 It has been seen (Art 263) that the pairs of straight lines passing through a given point S and conjugate to one another with respect to a given conic form an involution. Let a plane figure be given, containing a conic C, and let the figure homological with it be constructed, taking S as centre of homology, let C′ be the conic corresponding to C in the new figure. Since in two homological figures a harmonic pencil corresponds to a harmonic pencil, any pair of straight lines through S which are conjugate with respect to C will be conjugate also with respect to C′ The polars of S with respect . two conics will be corresponding straight lines, if then polar of S with respect to C be taken as the vanishing in the first figure, the polar of S with respect to C′ will lie at infinity, i e the point S will be the centre of the conic C′

In this case therefore any two straight lines through S which are conjugate with respect to C will be a pair of conjugate diameters of C′ If S is external to C, the double rays of the involution formed by the conjugate lines through S are the tangents from S to C, and therefore the asymptotes of C′, which is in this case a hyperbola If S is internal to C, the involution has no double rays, and therefore C′ is an ellipse

We conclude then that *to every point S in the plane of a given conic* C *corresponds a conic* C′ *homological with* C *and having its centre at S, which conic* C′ *is a hyperbola or an ellipse according as S is external or internal to the given conic* C

* Steiner, *Vorlesungen uber synthetische Geometrie* (ed Schroter, II[ter] Abschnitt, § 35, Zech, *Hohere Geometrie* (Stuttgart, 1857), § 7, Reye, *Geometrie der Lage* (2nd ed, Hannover, 1877), *Vortrag* 13

If the involution has no double points, each of the two points (Art 128) at which the pairs PP' subtend a right angle will be a focus of the conic For every pair of mutually perpendicular straight lines which meet in such a point will pass through two points P, P', and will therefore be conjugate lines with respect to the conic

From this it follows that one at least of the two axes of a conic contains two foci Further, a conic has only two foci, for every straight line which joins two foci is an axis (Art 341), and no conic (except it be a circle) has more than two axes

Consequently *a central conic (ellipse or hyperbola) has two foci, which are the double points of the involution PP' on an axis and are also the points at which the pairs of points PP' of the involution on the other axis subtend a right angle*

The axis which contains the foci may on this account be called the *focal axis* Since the foci are internal points, it is seen that in the hyperbola the focal axis is that one which cuts the curve (the transverse axis)

Since the centre O of the conic is the centre of the involution PP', it bisects the distance between the two foci

From what has been said it follows that *two perpendicular straight lines which are conjugate with respect to a conic meet the focal axis in two points which are harmonically conjugate with respect to the foci, and they determine upon the other axis a segment which subtends a right angle at either focus*

344 The *normal* at any point on a curve is the perpendicular at this point to the tangent Since the tangent and normal at any point on a conic are conjugate lines at right angles, they meet the focal axis in a pair of points harmonically conjugate with respect to the foci, and they determine on the other axis a segment which subtends a right angle at either focus (Art 343) Accordingly

If a circle be drawn to pass through the two foci and through any point on the conic, it will have the two points in which the non-focal axis is cut by the tangent and normal at that point as extremities of a diameter

And again (Art 60)

The tangent and normal at any point on a conic are the bisectors

of the angle made with one another by the rays which join that point to the foci *

These rays are called the *focal radii* of the given point.

345 A pair of conjugate lines which intersect at right angles in a point S external to the conic are harmonically conjugate with respect to the tangents from S to the conic (Art 264) as well as with respect to the rays joining S to the foci (Art 343), therefore

The angle between two tangents and that included by the straight lines which join the point of intersection of the tangents to the foci have the same bisectors †

346 In the parabola, the point at infinity on the axis, regarded as a point P, coincides with its correspondent P', for the straight line at infinity, being a tangent to the conic at the said point P, passes through its own pole

Accordingly one of the double points of the involution determined on the axis by the pairs of conjugate orthogonal rays, *i e* one of the foci, is at infinity The other [...] lies at a finite distance, and is generally spoken of a[...] of the parabola

Since in the case of the parabola one focus is at [...] the theorems proved above (Arts 343–345) becom[...] following

Two conjugate orthogonal rays, and in particular the tangent and normal at any point on the parabola, meet the axis in two points which are equidistant from the focus

The tangent and normal at a point on a parabola are the bisectors of the angle which the focal radius of the point makes with the diameter passing through the point ‡

The straight line which connects the focus with the point of intersection of two tangents to a parabola makes with either of the tangents the same angle that the axis makes with the other tangent

347 From the last of these may be immediately deduced the following theorem

The circle circumscribing a triangle formed by three tangents to a parabola passes through the focus

Let PQR (Fig 212) be a triangle formed by three

* APOLLONIUS, *loc cit*, III 48

† Ibid III 46

‡ DE LA HIRE, *loc cit*, lib VIII prop 2

tangents to a parabola, and let F be the focus Considering the tangents which meet in P, the angle FPQ is equal to that made by PR with the axis, and considering the tangents which meet in R, the angle FRQ is equal to that made by RP with the axis Hence the angles FPQ, FRQ are equal, and therefore P, Q, R, F lie on the same circle

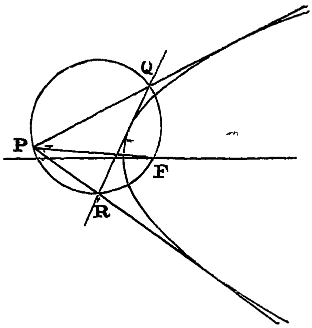

Fig 212

COROLLARY *The locus of the foci of all parabolas which touch the three sides of a given triangle is the circumscribing circle of the triangle*

This corollary gives the construction for the focus of a parabola which touches four given straight lines And since only one such parabola can be drawn (Art 157), we conclude that

Given four straight lines, the circles circumscribing the four triangles which can be formed by taking the lines three and three together all pass through the same point

348 The polar of a focus is called a *directrix*

The two directrices are straight lines perpendicular to the transverse axis and external to the conic, since the foci lie on the transverse axis and are internal to the conic (Art 343)

In the case of the parabola, the straight line at infinity is one directrix, the other lies at a finite distance, and is generally spoken of as *the* directrix of the parabola

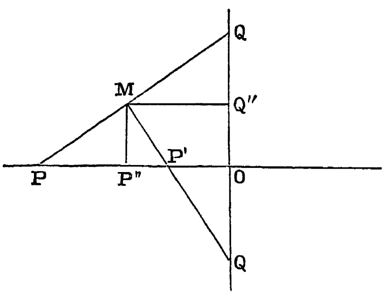

Fig 213

If F be a focus, and if the tangent at any point X on a conic cut the corresponding directrix in Y, this point Y will be the pole of the focal radius FX Therefore FX, FY are conjugate lines with respect to the conic, and since they meet in a focus, they will be at right angles consequently

The part of a tangent to a conic intercepted between its point of contact and a directrix subtends a right angle at the corresponding focus

349 Let the tangent and normal at any point M on a conic meet the focal axis in P, P' respectively, and let them meet the other axis in Q, Q' respectively (Fig 213) From M let perpendiculars MP'', MQ'' be drawn to the axes

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From the similar triangles OPQ, $Q''MQ$

$$OP \quad OQ = Q''M \quad Q''Q,$$

and from the right-angled triangle $Q'MQ$

$$Q''M \quad Q''Q = Q'Q'' \quad Q''M,$$

$$OP \quad OQ = Q'Q'' \quad Q''M$$

$$= Q'Q'' \quad OP'',$$

or

$$OP \quad OP'' = OQ \quad Q'Q''$$

$$= OQ\,(Q'O + OQ''),$$

so that

$$OP \quad OP'' - OQ \quad OQ'' = OQ \quad Q'O \qquad (1)$$

But P and P'' are a pair of conjugate points, since MP'' is the polar of P, similarly Q and Q'' are conjugate points Therefore (Art 294)

$$OP \quad OP'' = OA^2 \text{ and } OQ \quad OQ'' = \pm OB^2,$$

where OA, OB are the lengths of the semiaxes, and the double sign refers to the two cases of the ellipse and the hyperbola Again, the points Q, Q' subtend a right angle at either of the two foci F, F' (Art 343) so that

$$OQ \quad Q'O = OF^2$$

Substituting, (1) becomes

$$OF^2 = OA^2 \mp OB^2$$

This shows that in the ellipse $OA > OB$, so that the focal axis is the axis major

Referring now to Figs 214 and 215,

$$FA = FO + OA,$$

$$FA' = FO + OA' = FO - OA,$$

$$FA \quad FA' = FO^2 - OA^2$$

$$= \mp OB^2$$

If D be the point in which a directrix cuts the focal axis, the vertices A and A' of the conic will be harmonically conjugate with respect to F and the point D where the polar of F cuts AA' (Art 264), therefore, since O bisects AA',

$$OA^2 = OF \quad OD$$

tangents to a parabola, and let F be the focus Considering the tangents which meet in P, the angle FPQ is equal to that made by PR with the axis, and considering the tangents which meet in R, the angle FRQ is equal to that made by RP with the axis Hence the angles FPQ, FRQ are equal, and therefore P, Q, R, F lie on the same circle

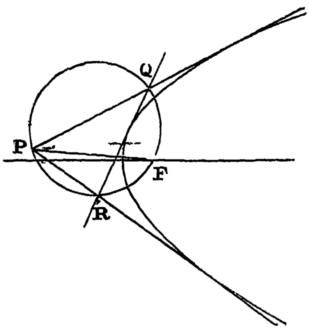

Fig 212

Corollary *The locus of the foci of all parabolas which touch the three sides of a given triangle is the circumscribing circle of the triangle*

This corollary gives the construction for the focus of a parabola which touches four given straight lines And since only one such parabola can be drawn (Art 157), we conclude that

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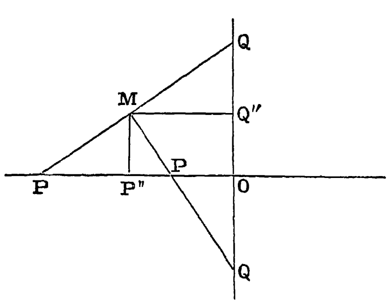

Fig 213

If F be a focus, and if the tangent at any point X on a conic cut the corresponding directrix in Y, this point Y will be the pole of the focal radius FX Therefore FX, FY are conjugate lines with respect to the conic, and since they meet in a focus, they will be at right angles consequently

The part of a tangent to a conic intercepted between its point of contact and a directrix subtends a right angle at the corresponding focus

349 Let the tangent and normal at any point M on a conic meet the focal axis in P, P' respectively, and let them meet the other axis in Q, Q' respectively (Fig 213) From M let perpendiculars MP'', MQ'' be drawn to the axes

read [illegible] respectively

From the similar triangles OPQ, $Q''MQ$

$$OP \cdot OQ = Q''M \cdot Q''Q,$$

and from the right-angled triangle $Q'MQ$

$$\begin{aligned} Q''M \cdot Q''Q &= Q'Q'' \cdot Q''M, \\ OP \cdot OQ &= Q'Q'' \cdot Q''M \\ &= Q'Q'' \cdot OP'', \end{aligned}$$

or

$$\begin{aligned} OP \cdot OP'' &= OQ \cdot Q'Q'' \\ &= OQ\,(Q'O + OQ''), \end{aligned}$$

so that

$$OP \cdot OP'' - OQ \cdot OQ'' = OQ \cdot Q'O \qquad (1)$$

But P and P'' are a pair of conjugate points, since MP'' is the polar of P, similarly Q and Q'' are conjugate points Therefore (Art 294)

$$OP \cdot OP'' = OA^2 \text{ and } OQ \cdot OQ'' = \pm OB^2,$$

where OA, OB are the lengths of the semiaxes, and the double sign refers to the two cases of the ellipse and the hyperbola Again, the points Q, Q' subtend a right angle at either of the two foci F, F' (Art 343) so that

$$OQ \cdot Q'O = OF^2$$

Substituting, (1) becomes

$$OF^2 = OA^2 \mp OB^2$$

This shows that in the ellipse $OA > OB$, so that the focal axis is the axis major

Referring now to Figs 214 and 215,

$$\begin{aligned} FA &= FO + OA, \\ FA' &= FO + OA' = FO - OA, \\ FA \cdot FA' &= FO^2 - OA^2 \\ &= \mp OB^2 \end{aligned}$$

If D be the point in which a directrix cuts the focal axis, the vertices A and A' of the conic will be harmonically conjugate with respect to F and the point D where the polar of F cuts AA' (Art 264), therefore, since O bisects AA',

$$OA^2 = OF \cdot OD$$

The parabola has one vertex at infinity, consequently the other lies midway between the focus and the directrix (Fig 218)

350 If a focus F of a conic C be taken as centre of homology, and a conic C′ be constructed homological with C and

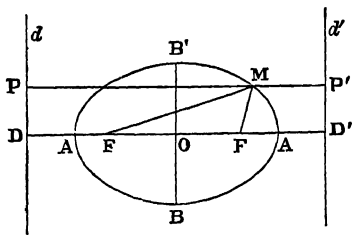

Fig 214

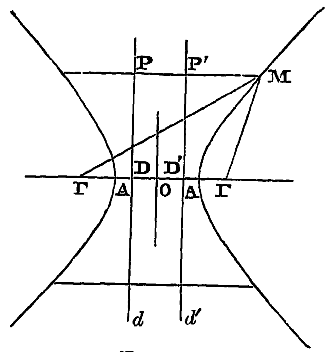

Fig 215

having its centre at F, it has been seen (Arts 340, 341) that C′ is a circle But by what has been proved in Art 77, if M and M' are a pair of corresponding points of C and C′,

$$\frac{FM}{FM'} \cdot MP = \text{constant},$$

or

$$\frac{FM}{MP} = FM' \times \text{constant},$$

where MP (Figs 214, 215) is the distance of M from the vanishing line, that is from the polar of F, *i e* the corresponding directrix Now FM' is constant, because C′ is a circle, therefore

The distance of any point on a conic from a focus bears a constant ratio to its distance from the corresponding directrix

Moreover, *this ratio is the same for the two foci* For let O (Figs 214, 215) be the centre of the conic, F, F' the foci, A, A' the vertices lying on the focal axis, D, D' the points in which this axis is cut by the directrices, then (Art 291)

$$OA^2 = OA'^2 = OF \cdot OD = OF' \cdot OD'$$

But $OF' = -OF$, so that $A'D' = -AD$ and $F'A' = -FA$, and therefore $FA \cdot AD = F'A' \cdot A'D'$,

which shows that the ratio is the same for F and for F'

In the case of the parabola the ratio in question is unity,

because (Art 349) the vertex of a parabola is equally distant from the focus and the directrix Therefore

The distance of any point on a parabola from the focus is equal to its distance from the directrix

351 Conversely, *the locus of a point M which is such that its distance from a fixed point F bears a constant ratio ϵ to its distance from a fixed straight line d is a conic of which F is a focus and d the corresponding directrix**

For let MP (Figs 214, 215) be drawn perpen
then by hypothesis

$$\frac{FM}{MP} = \epsilon$$

Let now the figure be constructed which is homological with the locus of M, F being taken as centre of homology, and d as vanishing line If M' be the point corresponding to M, then (Art 77)

$$\frac{FM}{FM'} \, MP = \text{constant}$$

These two equations show that FM' is constant, thus the locus of M' is a circle, centre F The locus of M is therefore a conic (Art 23) having one focus at F (Art 341) And since the straight line at infinity is the polar of F with respect to the circle, the straight line d is the polar of F with respect to the conic, i e it is the directrix corresponding to F

352 The length of a chord of a conic drawn through a focus perpendicular to the focal axis is called the *latus rectum* or the *parameter* of the conic

Let MFM' (Fig 216) be a chord of a conic drawn through a focus F and let N be the point where it cuts the corresponding directrix Let LFL' be the latus rectum drawn through F Then since the directrix is the polar of the focus, N and F are harmonic conjugates with regard to M and M' Therefore

$$\frac{2}{NF} = \frac{1}{NM} + \frac{1}{NM'},$$

and if perpendiculars MK, FD, $M'K'$ be let fall on the directrix,

$$\frac{2}{FD} = \frac{1}{M'K'} + \frac{1}{MK}$$

* Pappus, *Math Collect*, lib vii prop 238

But by Art 350

$$M'K' \cdot FD \cdot MK = M'F \cdot FL \cdot FM,$$

$$\frac{2}{FL} = \frac{1}{M'F} + \frac{1}{FM},$$

that is to say

In any conic, half the latus rectum is a harmonic mean between the segments of any focal chord

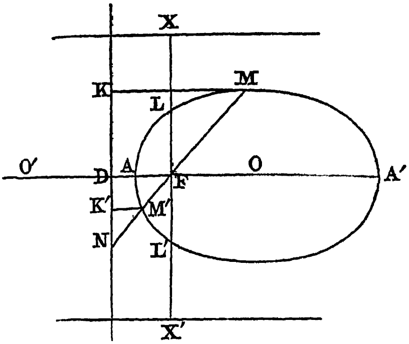

Fig 216

COROLLARY If M, M' be taken at A', A respectively,

$$\frac{1}{FL} = \frac{1}{2}\left(\frac{1}{AF} + \frac{1}{FA'}\right)$$

$$= \frac{1}{2}\frac{AA'}{AF \cdot FA'}$$

$$= \frac{OA}{\pm OB^2} \text{ (by Art 349)},$$

so that $$FL = \pm \frac{OB^2}{OA},$$

which gives the length of the semi-latus rectum in terms of the semi-axes

In the parabola $\frac{1}{FA'} = 0$, so that $FL = 2FA$

353 THEOREM *In the ellipse the sum, and in the hyperbola the difference, of the focal radii of any point on the curve is constant* *

Let M be any point on a central conic (Figs 214, 215) whose

* APOLLONIUS, *loc cit*, III 51, 52

foci are F, F' and directrices d, d', and let (M, d) &c denote as usual the distance of M from d, &c By Art 351

$$\frac{FM}{(M,d)} = \frac{F'M}{(M,d')} = \epsilon,$$

$$\frac{FM \pm F'M}{(M,d) \pm (M,d')} = \epsilon$$

But (Fig 214) in the ellipse $(M, d) + (M, d')$, and (Fig 215) in the hyperbola $(M, d) - (M, d')$ is equal to the distance DD' between the two directrices, therefore

$$FM \pm F'M = \epsilon \cdot DD',$$

which proves the proposition

Conversely *The locus of a point the sum (difference) of whose distances from two fixed points is constant is an ellipse (a hyperbola) of which the given points are the foci*

354 If in the proposition of the last Article the point M be taken at a vertex A,

$$\begin{aligned} \epsilon \cdot DD' &= FA \pm F'A \\ &= 2OA \\ &= AA', \end{aligned}$$

so that *the length of the focal axis is the constant value of the sum or difference of the focal radii* It is seen also that the constant ϵ is equal to the ratio of the length of the focal axis to the distance between the directrices

355 Since by Art 294

$$OA^2 = OF \cdot OD,$$

or

$$AA'^2 = FF' \cdot DD',$$

$$\epsilon = \frac{AA'}{DD'} = \frac{FF'}{AA'},$$

so that the constant ϵ is equal to the ratio of the distance between the foci to the length of the focal axis Now in the ellipse $FF' < AA'$, in the hyperbola $FF' > AA'$, in the parabola $FF' = AA' = \infty$, in the circle $FF' = 0$ Therefore the conic is an ellipse, a hyperbola, a parabola, or a circle, according as $\epsilon < 1$, $\epsilon > 1$, $\epsilon = 1$, or $\epsilon = 0$ This constant ϵ is called the *eccentricity* of the conic

356 THEOREM *The locus of the feet of perpendiculars let fall from a focus upon the tangents to an ellipse or hyperbola is the circle described on the focal axis as diameter**

* APOLLONIUS, *loc cit*, III 49, 50

Take the case of the ellipse (Fig. 217) If F, F' are the foci, and M is any point on the curve, join $F'M$ and produce it to G making MG equal to MF Then $F'G$ will (Art 354) be equal to AA' whatever be the position of M, thus the locus of G is a circle, centre F' and radius equal to AA'

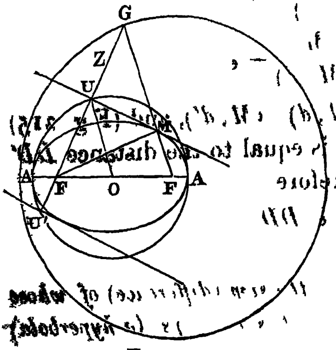

Fig 217

If FG be joined, it will cut the tangent at M perpendicularly, since this tangent (Art 344) bisects the angle FMG, and the point U where the two lines intersect will be the middle point of FG because FMG is an isosceles triangle Therefore OU is parallel to $F'G$ and equal to $\frac{1}{2}F'G$, that is, to OA, i e the locus of U is the circle on AA' as diameter

A similar proof holds good for the hyperbola, except that from the greater of the two MF, MF' must be *cut off* a part MG equal to the less

357 If FU, FU' (Fig 217) are the perpendiculars let fall from a focus F on a pair of parallel tangents, U, F, U' will evidently be collinear And since U and U' both lie on the circle described on AA' as diameter,

$$FU \cdot FU' = FA \cdot FA'$$
$$= \mp OB^2 \text{ (Art 349)},$$

according as the conic is an ellipse or a hyperbola

Thus *the product of the distances of a pair of parallel tangents from a focus is constant*

Since the perpendicular let fall from the other focus F' on the tangent at M is equal to FU', it follows that

The product of the distances of any tangent to an ellipse (hyperbola) from the two foci is constant, and equal to the square of half the minor (conjugate) axis

Conversely *The envelope of a straight line which moves in such a way that the product of its distances from two fixed points is constant is a conic, an ellipse if the value of the constant is positive, a hyperbola if it is negative*

358 Let F (Fig 218) be the focus of a parabola, A the vertex, M any point on the curve, N the point of intersection of the tangents at M and A If NF' be drawn to the infinitely

distant focus F' (i e if NF' be drawn parallel to the axis), the angles ANF', FNM will be equal (Art 346) But ANF' is a right angle, therefore FNM is a right angle also Thus

The foot of the perpendicular let fall from the focus of a parabola on any tangent lies on the tangent at the vertex

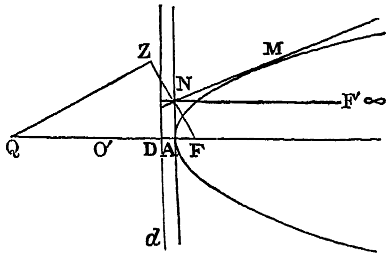

Fig 218

COROLLARY Since any point on the circumscribing circle of a triangle may be regarded (Art 347) as the focus of a parabola inscribed in the triangle, it follows at once from the theorem just proved that *if from any point on the circumscribing circle of a triangle perpendiculars be let fall on th three sides, their feet will be collinear* *

359 The theorem of Art 356 may be put into the following form

If a right angle move in its plane in such a way that its vertex describes a fixed circle, while one of its arms passes always through a fixed point, the envelope of its other arm will be a conic concentric with the given circle, and having one focus at the fixed point The conic is an ellipse or a hyperbola according as the given point lies within or without the given circle †

So too the corresponding theorem (Art 358) for the parabola may be expressed in a similar form as follows

If a right angle move in its plane in such a way that its vertex describes a fixed straight line while one of its arms passes always through a fixed point, the other arm will envelope a parabola having the fixed point for focus and the fixed straight line for tangent at its vertex

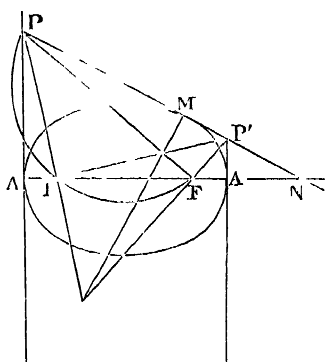

Fig 19

360 I Let the tangents at the vertices of a central conic be cut in P, P' by the tangent at any point M (Fig 219) The three tangents form a triangle circumscribed about the conic, two of the vertices of which are P and P', the third (at infinity) being the pole of the

* For other proofs of this see Art 416

† MACLAURIN, *Geometria Organica*, pars II[a] prop XI

axis AA' Therefore (Art 274) the straight lines drawn from P and P' to any point on the axis will be conjugate to one another with respect to the conic Thus, in particular, the straight lines joining P and P' to a focus will be conjugate to one another, but conjugate lines which meet in a focus are mutually perpendicular (Art 343), consequently the circle on PP' as diameter will cut the axis AA' at the foci *

II Let the tangent PMP' cut the axis AA' at N, then N is the harmonic conjugate of M with respect to P, P' (Art 194)

Consider now the complete quadrilateral formed by the lines $FP, F'P, FP', F'P'$ Two of its diagonals are FF' and PP', the third diagonal must then cut FF' and PP' in points which are harmonically conjugate to N with regard to F, F' and P, P' respectively It must therefore be the normal at M to the conic †

361 Let TM, TN (Fig 220) be a pair of tangents to a conic, M and N their points of contact, F a focus, d the corresponding directrix If the chord MN cut d in P, this point is the pole of TF, therefore TFP is a right angle (Art 343) ‡

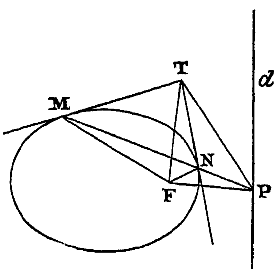

Fig 220

But MN is divided harmonically by FT and its pole P, thus $F(MNTP)$ is a harmonic pencil, and consequently FT, FP are the bisectors of the angle MFN Accordingly

One of the bisectors of the angle which a chord of a conic subtends at a focus passes through the pole of the chord The other bisector meets the chord at its point of intersection with the directrix corresponding to the focus

Or the same thing may be stated in a different manner, thus

The straight line which joins a focus to the point of intersection of a pair of tangents to a conic makes equal (or supplementary) angles with the focal radii of their points of contact §

* APOLLONIUS, *loc cit*, III 45, DESARGUES, *Œuvres*, I pp 209 210

† APOLLONIUS, *loc cit*, III 47

‡ If the points M and N are taken indefinitely near to one another this reduces to the theorem already proved in Art 348

§ DE LA HIRE, *loc cit*, lib VIII prop 24

362 Let the tangents TM, TN be cut by any third tangent in M', N' respectively (Figs 221, 222), let L be the point of contact of this third tangent The following relations will hold among the angles of the figures

$$N'FL = NFN' = \tfrac{1}{2}NFL,$$
$$LFM' = M'FM = \tfrac{1}{2}LFM,$$

whence by addition

$$N'FL + LFM' = \tfrac{1}{2}(NFL + LFM),$$

or

$$N'FM' = \tfrac{1}{2}NFM = NFT = TFM\,*$$

Let now the tangents TM, TN be fixed, while the tangent $M'N'$ is supposed to vary By what has just been proved, the angle subtended at the focus by the part $M'N'$ of the

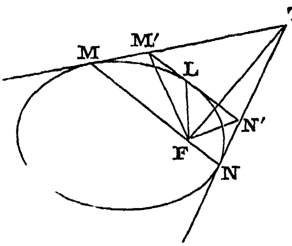

Fig 221

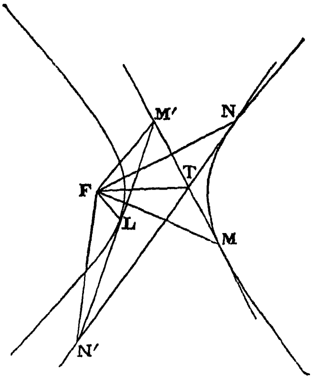

Fig 222

variable tangent intercepted between the two fixed ones is constant As the variable tangent moves, the points M', N' describe two projective ranges (Art 149), and the arms FM', FN' of the constant angle $M'FN'$ trace out two concentric and directly equal pencils (Art 108) Accordingly

* In this reasoning it is supposed that FM', FN', FL are all internal bisectors, i e that either the conic is an ellipse or a parabola, or that if it is a hyperbola, the three tangents all touch the same branch (Fig 221) If on the contrary two of the tangents for example TM and TN touch one branch and the third $M'N'$ the other branch (Fig 222), then FM' and FN' will be external bisectors In that case

$$N'FL = \tfrac{1}{2}NFL - \frac{\pi}{2}$$

$$LFM' = \tfrac{1}{2}LFM + \frac{\pi}{2}$$

(the angles being measured all in the same direction),

$$N'FM' = \tfrac{1}{2}NFM, \text{ just as in the case above}$$

The ranges which a variable tangent to a conic determines on two fixed tangents are projected from either focus by means of two directly equal pencils

This theorem clearly holds good for the cases of the parabola and its infinitely distant focus, and the circle and its centre For the parabola it becomes the following

Two fixed tangents to a parabola intercept on any variable tangent to the same a segment whose projection on a line perpendicular to the axis is of constant length.

The general theorem may also be put into the following form

One vertex F of a variable triangle $M'FN'$ is fixed, and the angle $M'FN'$ is constant, while the other vertices M', N' move respectively on fixed straight lines TM, TN The envelope of the side $M'N'$ is a conic of which F is a focus, and which touches the given lines TM, TN

363. The problem, *Given the two foci F, F' of a conic and a tangent t, to construct the conic*, is determinate, and admits of a single solution, as follows

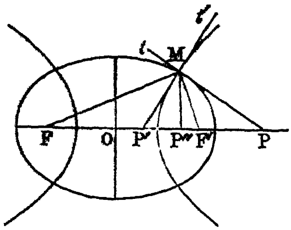

Fig 223

Join FF' (Figs 223, 224) and let it cut t in P, take P' the harmonic conjugate of P with respect to F and F' If a straight line $P'M$ be drawn perpendicular to t, it will be the normal corresponding to the tangent t (Art 344), *i e* M will be the point of contact of t Draw MP'' perpendicular to FF', it will be the polar of P, and P, P'' will be conjugate points with respect to the

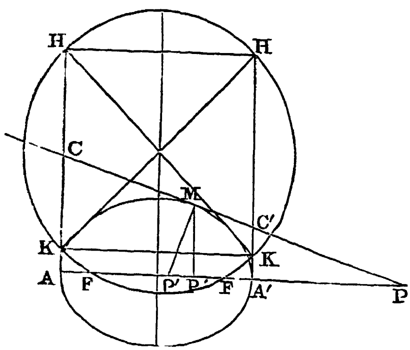

Fig 224

conic If then FF' be bisected at O, and on FF' there be taken two points A A' such that $OA^2 = OA'^2 = OP \cdot OP''$, A and A' will

be the vertices of the conic The conic is therefore completely determined, for three points on it are known (M, A, A') and the tangents at these three points (t and the straight lines AC, $A'C'$ drawn through A, A' at right angles to AA')

An easy method of constructing the conic by tangents is to describe any circle through F and F', cutting AC, $A'C'$ in H and K, H' and K' respectively (Fig 224) Then if the chords HK', $H'K$ be drawn which intersect crosswise in the centre of the circle (which lies on the non-focal axis), these will be tangents to the conic (Art 360) Every circle through F and F' which cuts AC and $A'C'$ thus determines two tangents to the conic

The conic is an ellipse or a hyperbola according as t cuts the segment FF' externally or internally

The conic is a parabola when F' is at infinity (Fig 225) In this case produce the axis PF to P' making FP' equal to PF, and draw $P'M$ perpendicular to t, then M will be the point of contact of the given tangent t Draw MP'' perpendicular to the axis, then P and P'' will be conjugate points with respect to the parabola And since the involution of conjugate points on the axis has one double point at infinity, the middle point A of PP'' will be the other double point, *i e* the vertex of the parabola The parabola is therefore completely determined, since two points on it are known (M and A), and the tangents at these points (t and the straight line drawn through A at right angles to the axis)

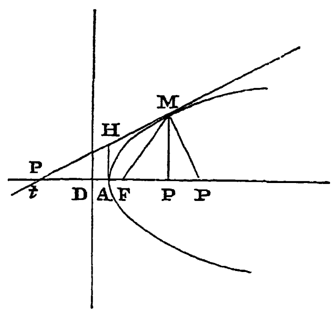

Fig 225

364 On the other hand, the problem, *To construct the conic which has its foci at two given points F, F' and which passes through a given point M,* which is also a determinate one, admits of two solutions For if the locus of a point be sought the *sum* of whose distances from F and F' is equal to the constant value $FM + F'M$, an *ellipse* is arrived at, but if the locus of a point be sought the *difference* of whose distances from F and F' is equal to $FM \backsim FM'$, a *hyperbola* is found

This may also be seen from the theorem of Art 344, which shows that if the straight lines t, t' be drawn bisecting the angle FMF' (Fig 223) each of these lines will be a tangent at M to a conic which satisfies the problem, the other line being the corresponding normal to this conic The finite segment FF' is cut or not by the tangents according as the conic is a hyperbola or an ellipse There will consequently be two conics which have F, F' for foci and which pass

through M, a hyperbola having for tangent at M that bisector t' which cuts the segment FF', and for normal the other bisector t, and an ellipse having t for tangent at M and t' for normal

These two conics, having the same foci, are concentric and have their axes parallel They will cut one another in three other points besides M, and their four points of intersection will form a rectangle inscribed in the circle of centre O and radius OM, in other words, the three other points will be symmetrical to M with respect to the two axes and the centre. This is evident from the fact that a conic is symmetrical with respect to each of its axes

365. Through every point M in the plane then pass two conics, an ellipse and a hyperbola, having their foci at F and F' In other words, the system of *confocal conics* having their foci at F and F' is composed of an infinity of ellipses and an infinity of hyperbolas, and through every point in the plane pass one ellipse and one hyperbola, which cut one another there orthogonally and intersect in three other points.

Two conics of the system which are of the same kind (both ellipses or both hyperbolas) clearly do not intersect at all

Two conics of the system however which are of opposite kinds (one an ellipse, the other a hyperbola) always intersect in four points, l cut one another orthogonally at each of them This may be seen by observing that the vertices of the hyperbola are points lying within the segment FF', and therefore within the ellipse On the other hand, there must be points on the hyperbola which lie outside the ellipse, for the latter is a closed curve which has all its points at a finite distance, while the former extends in two directions to infinity The hyperbola therefore, in passing from the inside to the outside of the ellipse, must necessarily cut it

No two conics of the system can have a common tangent, because (Art 363) only one conic can be drawn to have its foci at given points and to touch a given straight line

Any straight line in the plane will touch a determinate conic of the system, and will be normal, at the same point, to another conic of the system, belonging to the opposite kind The first of these conics is a hyperbola or an ellipse according as the given straight line does or does not cut the finite segment FF'

366 If first point F' lies at infinity, the problem of Art 364 becomes the following *Given the axis of a parabola, the focus F, and a point M on the curve, to construct the parabola*

Just as in Art 364, there are two solutions (Fig 226) The tangents at M to the two parabolas which satisfy the problem are the bisectors of the angle made by MF with the diameter passing through M, therefore the parabolas cut orthogonally at M and

consequently intersect at another point, symmetrical to M with respect to the axis The parabolas cannot intersect in any other finite point, since they touch one another at infinity *

The tangents to the two parabolas at M cut the axis in two points P, P' which lie at equal distances on opposite sides of F, and if P'' is the foot of the perpendicular let fall from M on the axis, the vertices A, A' of the parabolas are the middle points of the segments PP'', $P'P''$ respectively

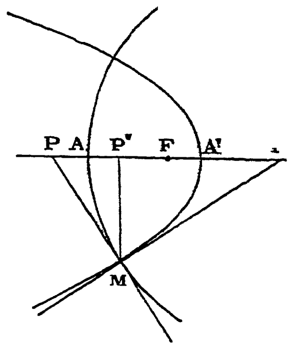

Fig 226

Suppose A and P'' to fall on the same side of F Then since $P'P''<P'P$, and $P'A'$ is the half of $P'P''$, and $P'F$ the half of $P'P$, therefore $P'A'<P'F$, i e A and A' fall on opposite sides of F It follows that in the system composed of the infinity of parabolas which have a common axis and focus, two parabolas intersect (orthogonally and in two points) or do not intersect, according as their vertices lie on opposite sides or on the same side of the common focus

Since F, A, A' are the middle points of PP', PP'', $P'P''$ respectively, we have the relations

$$\begin{aligned} FP+FP' &= 0, \\ 2FA &= FP+FP'', \\ 2FA' &= FP'+FP'', \end{aligned}$$

whence the following are easily deduced

$$\begin{aligned} FP'' &= FA+FA',\dagger \\ FP &= FA-FA'=A'A, \\ FP' &= FA'-FA = AA' \end{aligned}$$

These last relations enable us at once to find the points P, P', P'' when A and A' are known The point M (and the symmetrical point in which the parabolas intersect again) can then be constructed by observing that FM is equal to FP or FP'

367 It has been seen that a conic is determined when the two foci and a tangent are given It can also be shown that *a conic is determined when one focus and three tangents are given*, this follows

* That is to say, if the figure be constructed which is homological with that formed by the two parabolas it will consist of two conics touching one another at a point situated on the vanishing line of the new figure, and intersecting in two other points

† Hence the middle point of AA' is also the middle point of FP''

follows from the proposition at the end of Art 362 For let LMN (Fig. 227) be the triangle formed by the three given tangents, and F the given focus. Then the conic is seen to be the envelope of the base $M'N'$ of a variable triangle $M'FN'$, which is such that the vertex F is fixed, the angle $M'FN'$ is always equal to the constant angle MFN, and the vertices M', N' move on the fixed straight lines LM, LN respectively

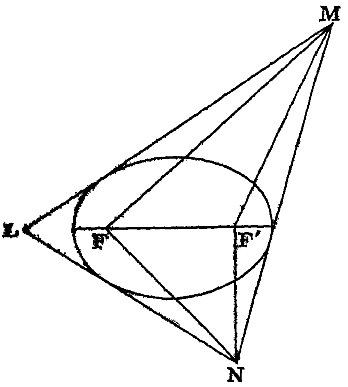
Fig 227

In order to determine the other focus F', we make use of the theorem of Art 345 At the point M make the angle LMF' equal to FMN, and at the point N make the angle LNF' equal to FNM (all these angles being measured in the same direction), then the point of intersection of MF', NF' will be the second focus F'

The investigation of the circumstances under which the conic is an ellipse, a hyperbola, or a parabola, is left as an exercise to the student The following are the results

(1) The conic is an ellipse if F lies within the triangle LMN, or if F lies without the circle circumscribing the triangle LMN and within one of the (infinite) spaces bounded by one of the sides of the triangle and the other two produced

(2) a hyperbola if F lies inside the circle but outside the triangle, or if it lies within one of the (infinite) V-shaped spaces which have one of the angular points of the triangle LMN for vertex and are bounded by the sides meeting in that angular point, both produced backwards

(3) a parabola if F lies on the circle circumscribing the triangle LMN, as we have seen already (Art 347) *

368 Let TU, TV (Fig 228) be a pair of tangents to an ellipse or hyperbola which intersect at right angles If perpendiculars FU, $F'U'$ and FV, $F'V'$ be let fall upon them respectively from the foci F and F', then evidently $TU = VF$ and $TU' = V'F'$ But by Art 357 we have $VF \cdot V'F' = \pm OB^2$, therefore $TU \cdot TU' = \pm OB^2$ But since U and U' both lie on

* STEINER, *Développement d'une série de theorèmes relatifs aux sections coniques* (Annales de Gergonne, t xix 1828, p 47), Collected Works, vol 1 p 198

the circle described upon the focal axis AA' as diameter (Art 356), the rectangle $TU \cdot TU'$ is the *power* of the point T with respect to this circle, and is equal to $OT^2 - OA^2$ Thus

$$OT^2 = OA^2 \pm OB^2 = \text{constant},$$

so that we have the following theorem *

The locus of the point of intersection of two tangents to an ellipse or a hyperbola which cut at right angles is a concentric circle

This circle is called the *director circle* of the conic †

In the ellipse $OT^2 = OA^2 + OB^2$, so that the director circle circumscribes the rectangle formed by the tangents at the extremities of the major and minor axes In the hyperbola $OT^2 = OA^2 - OB^2$, so that pairs of mutually perpendicular tangents exist only if $OA > OB$ If $OA = OB$, i e if the hyperbola is equilateral (Art 395), the director circle reduces simply to the centre O, that is, the asymptotes are the only pair of tangents which cut at right angles If $OA < OB$, the director circle has no real existence, the hyperbola has no pair of mutually perpendicular tangents

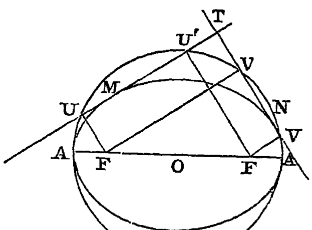

Fig 228

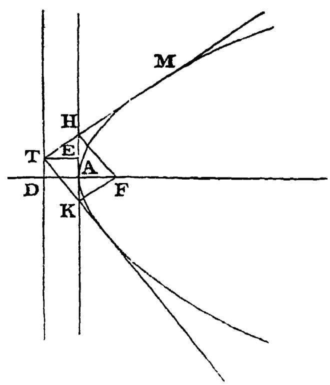

Fig 229

369 Consider now the case of the parabola (Fig 229) Let F be the focus, A the vertex, TH and TK a pair of mutually perpendicular tangents If these meet the tangent at the vertex in H and K respectively, the angles FHT, FKT will be right angles (Art 358), so that the figure $THFK$ is a rectangle Therefore $TH = KF$, and since the triangles TEH, FAK are evidently similar, $TE = AF$ The locus of the point T is

* De La Hire, *loc cit* lib viii props 27, 28

† Gaskin, *The geometrical construction of a conic section*, (Cambridge 1852) chap iii prop 10 et seqq

therefore a straight line parallel to HK, and lying at the same distance from HK (on the opposite side) that F does That is to say

The locus of the point of intersection of two tangents to a parabola which cut at right angles is the directrix *

Since the director circle of a conic is concentric with the latter, it must in the case of the parabola have an infinitely great radius. In other words, it must break up into the line at infinity and a finite straight line. And we have just seen that this finite straight line is the directrix.

370 The director circle possesses a property in relation to the self-conjugate triangles of the conic which we will now proceed to investigate Let XYZ (Fig 230) be a triangle which is self-conjugate with respect to a conic whose centre is O Join OX and let it cut YZ in X' and the conic in A' Draw OB' parallel to YZ, let it cut XY in L and the conic in B', and draw ZL' parallel to OX to meet OB' in L'

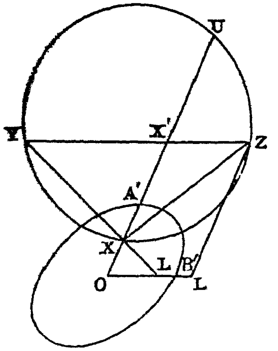

Fig 230

Then OA' and OB' are evidently conjugate semi-diameters, also X and X', L and L' are pairs of conjugate points with respect to the conic Therefore

$$OX \cdot OX' = \pm OA'^2, \text{ and } OL \cdot OL' = \pm OB'^2,$$

where the positive or the negative signs are to be taken according as the semidiameters OA', OB' are real or ideal (Art 294)

Thus for the ellipse

$$\begin{aligned} OX \cdot OX' + OL \cdot OL' &= OA'^2 + OB'^2 \\ &= OA^2 + OB^2, \end{aligned}$$

and for the hyperbola

$$\begin{aligned} OX \cdot OX' + OL \cdot OL' &= \pm(OA'^2 - OB'^2) \\ &= \pm(OA^2 - OB^2), \end{aligned}$$

so that in both cases (Art 368)

$$OX \cdot OX' + OL \cdot OL' = OT^2, \tag{1}$$

where OT is the radius of the director circle

* De la Hire, *loc cit*, lib viii prop 26

Now let a circle be described round the triangle XYZ, and let U be the point where it cuts OX again, then

$$X'Y \cdot X'Z = X'X \cdot X'U,$$

$$X'U = \frac{X'Y}{X'X} \cdot X'Z$$

$$= \frac{OL}{OX} \cdot X'Z,$$

(from the similar triangles OLX, $X'YX$)

$$= \frac{OL}{OX} \cdot OL'$$

Therefore equation (1) gives

$$OT^2 = OX \cdot OX' + OX \cdot X'U$$

$$= OX \cdot OU,$$

that is to say *The centre of a conic has with respect to the circumscribing circle of any triangle self-conjugate to the conic a constant power, which is equal to the square of the radius of the director circle*

Or in other words

*The circle circumscribing any triangle which is self-conjugate with regard to a conic is cut orthogonally by the director circle**

The following particular cases of this theorem are of interest

I *The centre of the circle circumscribing any triangle which is self-conjugate with respect to a parabola lies on the directrix*

II *The circle circumscribing any triangle which is self-conjugate with respect to an equilateral hyperbola passes through the centre of the conic*

371 Consider a quadrilateral circumscribed about a conic Since each of its diagonals is cut harmonically by the other two, the circle described on any one of the diagonals as diameter is cut orthogonally by the circle which circumscribes the diagonal triangle (Art 69) But the diagonal triangle is self conjugate with respect to the conic (Art 260), and therefore its circumscribing circle cuts orthogonally the director circle (Art 370) Consequently the director circle and the three circles described on the diagonals as diameters all cut orthogonally the circle circumscribing the diagonal triangle Now by Newton's theorem (Art 318) the centres of the four first-named circles are collinear, and circles whose centres are collinear and which all cut the same circle orthogonally have a common radical axis Therefore

The director circle of a conic, and the three circles described on

* GASKIN, *loc cit*, p 33

the diagonals of any circumscribed quadrilateral as diameters, are coaxal.

In the parabola the director circle reduces to the directrix and the straight line at infinity; in this case then the above theorem becomes the following:

If a quadrilateral is circumscribed about a parabola, the three circles described on the diagonals of the quadrilateral as diameters have the directrix for their common radical axis.

372. If in the theorem of Art. 371 the quadrilateral be supposed to be given, and the conic to vary, we arrive at the following theorem:

The director circles of all the conics inscribed in a given quadrilateral form a coaxal system, to which belong the three circles having as diameters the diagonals of the quadrilateral.

There is one circle of such a system which breaks up into two straight lines: that namely which degenerates into the radical axis and the straight line at infinity. Now the director circle breaks up into two straight lines—viz. the directrix and the line at infinity—in the case of a parabola (Art. 369). Therefore the common radical axis of the system of coaxial director circles is the directrix of the parabola which can be inscribed in the quadrilateral.

If the circles of the system do not intersect, there are two of them which degenerate into point-circles (the limiting points). Now the director circle degenerates into a point in the case of the equilateral hyperbola (Art. 368). Therefore when the circles do not cut one another, the two limiting points of the system are the centres of the two equilateral hyperbolas which can in this case be inscribed in the quadrilateral. If the circles do intersect, the system has no real limiting points; and in this case no equilateral hyperbola can be inscribed in the quadrilateral.

The circles which cut orthogonally the circles of a coaxial system form another coaxial system; if the first system has real limiting points, the second system has not, and *vice versa*. In order then *to inscribe an equilateral hyperbola in a given quadrilateral*, it is only necessary to describe circles on two of the diagonals of the quadrilateral as diameters, and then to draw two circles cutting the former two orthogonally. When the problem is possible, these two orthogonal circles will intersect; and their two points of intersection are the centres of the two equilateral hyperbolas which satisfy the conditions of the problem.

373. If five points are taken on a conic, five quadrangles may be formed by taking these points four and four together; and the diagonal triangles of these five quadrangles are each of them self-conjugate with respect to the conic. If the circumscribing circles of

these five diagonal triangles be drawn, they will give, when taken together in pairs, ten radical axes. These ten radical axes will all meet in the same point, viz the centre of the conic

374. Consider again a quadrilateral circumscribing a conic, let P and P', Q and Q', R and R' be its three pairs of opposite vertices. If these be joined to any arbitrary point S, and if moreover from this point S the tangents t, t' are drawn to the conic, it is known by the theorem correlative to that of Desargues (Art 183, right) that t and t', SP and SP', SQ and SQ', SR and SR' are in involution. Now let one of the sides of the quadrilateral (say $P'Q'R'$) be taken to be the straight line at infinity, so that the inscribed conic is a parabola, and let S be taken at the orthocentre (centre of perpendiculars) of the triangle PQR formed by the other three sides of the quadrilateral Then each of the three pairs of rays SP and SP', SQ and SQ', SR and SR' cut orthogonally, therefore the same will be the case with the fourth pair t and t' But tangents to a parabola which cut orthogonally intersect on the directrix (Art 369), therefore

The orthocentre of any triangle circumscribing a parabola lies on the directrix

375 If in the theorem of the last Article the triangle be supposed to be fixed, and the parabola to vary, we obtain the theorem

The directrices of all parabolas inscribed in a given triangle meet in the same point, viz the orthocentre of the triangle

Given a quadrilateral, one parabola (and only one) can always be inscribed in it By taking the sides of the quadrilateral three and three together, four triangles are obtained, and the four orthocentres of these triangles must all lie on the directrix of the parabola It follows that

Given four straight lines, the orthocentres of the four triangles formed by taking them three and three together are collinear

376 Let **C** be any given conic, and let **C**$'$ be its polar reciprocal with respect to an auxiliary conic **K** The particular case in which **K** is a circle whose centre coincides with a focus F of the conic **C** is of great interest, we shall now proceed to consider it

If r, r' be any two straight lines which are conjugate with respect to **C**, and if R, R' be their poles with respect to **K**, it is known (Art 323) that R, R' will be conjugate points with respect to **C**$'$ Consider now two such lines r, r' which pass through F, they will be at right angles since every pair of conjugate lines through a focus cut one another orthogonally

They will therefore be perpendicular diameters of the circle **K**, and their poles R, R' with respect to **K** will be the points at infinity on r', r respectively These points are conjugate with respect to **C**$'$, and the straight lines joining them to the centre of this conic are therefore a pair of conjugate diameters of **C**$'$, consequently two conjugate diameters of **C**$'$ are always mutually perpendicular This proves that **C**$'$ is a circle, *i e the polar reciprocal of a conic, with respect to a circle which has its centre at one of the foci, is a circle*

By taking the steps of the above reasoning in the opposite order, the converse proposition may be proved, viz

The polar reciprocal of a circle with respect to an auxiliary circle is a conic having one focus at the centre of the auxiliary circle.

As in Art. 323, it is seen that the conic is an ellipse, a hyperbola, or a parabola, according as the centre of the auxiliary circle lies within, without, or upon the other circle.

377 If d be the directrix of the conic **C** corresponding to the focus F, and if its pole be taken with respect to the circle **K**, this point will evidently be the centre of the circle **C**$'$ (Art 323)

The radius of the circle **C**$'$ may also easily be found For in Fig 216 let two points X, X' be taken in the latus rectum LFL' such that

$$FX\ FL = FX'\ FL' = k^2,$$

where k denotes the radius of the circle **K**, and let straight lines be drawn through X and X' perpendicular to XFX' These straight lines are evidently parallel tangents of the circle **C**$'$, and the distance XX' between them is therefore equal in length to the diameter of **C**$'$ But

$$\tfrac{1}{2} X'X = FX = \frac{k^2}{FL},$$

so that the radius of the circle **C**$'$ is equal to $\dfrac{k^2}{FL}$

The eccentricity e of the conic **C** may be expressed in a simple manner in terms of quantities depending upon the two circles **K** and **C**$'$ For if O' be the centre and ρ the

radius of the latter circle, it has been seen that the is the polar of O' with respect to **K**, therefore (Fig

$$FD \cdot FO' = k^2$$

But it has just been proved that

$$FL \cdot \rho = k^2,$$

therefore (Art 351), $\quad e = \dfrac{FL}{FD} = \dfrac{FO'}{\rho}$

378 The proposition of Art 376 may be proved in a different manner, so as to lead at once to the position and size of the circle **C'**

Take any point M on the (central) conic **C** (Fig 217), from the focus F draw FU perpendicular to the tangent at M, and on FU take a point Z such that $FZ \cdot FU = k^2$, k being as before the radius of the circle **K** Then the locus of Z is the polar reciprocal of **C** with respect to **K**

Now it is known (Arts 356, 357) that U lies on the circle on AA' as diameter, and that if UF cut this circle again at U'

$$FU \cdot FU' = \mp OB^2$$

Therefore $\qquad FZ \cdot FU' = k^2 \cdot \mp OB^2,$

which proves (Art 23 [6]) that the locus of Z is a circle whose centre O' lies on FO and divides it so that $FO' \cdot FO = k^2 \cdot OB^2$, and whose radius ρ is equal to $k^2 \cdot \dfrac{OA}{OB^2}$, that is, (Art 352 Cor) to $\dfrac{k^2}{FL}$ And again, since $OF \cdot OD = OA^2$ and $FD = FO + OD$, (Figs 214, 215),

$$FD \cdot FO = OF^2 - OA^2 = \mp OB^2 = k^2 \frac{FO}{FO'}$$

by what has just been proved

$$FO' \cdot FD = k^2,$$

i e O' is the pole of the directrix d with respect to **K**

In the particular case where $k = OB$, $\rho = OA$, that is to say

The polar reciprocal of an ellipse (hyperbola) with respect to a circle having its centre at a focus and its radius equal to half the minor (conjugate) axis is the circle described on the major (transverse) axis as diameter

379 In the case where **C** is a parabola, let M be any point on the curve (Fig 218), let fall FN perpendicular to the tangent at M, and take on FN a point Z such that $FZ \cdot FN = k^2$ Then,

as before, the locus of Z will be the polar reciprocal of **C** with respect to **K**. Draw ZQ perpendicular to ZF to cut the axis of the parabola in Q

Then a circle will evidently go round $QANZ$, so that

$$FA \cdot FQ = FN \cdot FZ = k^2,$$

therefore Q is a fixed point, and the locus of Z is the circle on QF as diameter If O' be the centre, ρ the radius of this circle,

$$FO' = \rho = \tfrac{1}{2}\frac{k^2}{FA}$$

In the particular case where k is equal to half the latus rectum, that is, to $2FA$, we have $\rho = k$, that is to say

The polar reciprocal of a parabola with respect to a circle having its centre at the focus and its radius equal to half the latus rectum is a circle of the same radius, having its centre at the point of intersection of the axis with the directrix

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277,

CHAPTER XXIV

COROLLARIES AND CONSTRUCTIONS

380 In the theorem of Art 275 suppose the vertices B and C of the inscribed triangle ABC (Fig 188) to be the points at infinity on a hyperbola, then S will be the centre of the curve, and the theorem will become the following

If from any point A on a hyperbola parallels be drawn to the asymptotes, they will meet any given diameter in two points F and G which are conjugate to one another with regard to the curve Or

If through two points lying on a diameter of a hyperbola, which are conjugate to one another with regard to the curve, parallels be drawn to the asymptotes, they will intersect on the curve

From this follows a method for *the construction of a hyperbola by points, having given the asymptotes and a point M on the curve*

On the straight line SM, which joins M to the point of intersection S of the asymptotes, take two conjugate points of the involution determined by having S for centre and M for a double point These points will be conjugate to one another with respect to the conic (Art 263), if then parallels to the asymptotes be drawn through them, the two vertices of the parallelogram so formed will be points on the hyperbola which is to be constructed

381 Let similarly the theorem of Art 274 be applied to the hyperbola, taking the sides b and c of the circumscribed triangle abc to be the asymptotes, it will then become the following

If through the points where the asymptotes are cut by any tangent to a hyperbola any two parallel straight lines be drawn, these will be conjugate to one another with respect to the conic Or

Two parallel straight lines which are conjugate to one another with respect to a hyperbola cut the asymptotes in points, the straight lines joining which are tangents to the curve

From this we deduce a method for *the construction, by means of its tangents, of a hyperbola, having given the asymptotes b and c and one tangent m*

Draw parallel to m two conjugate rays of the involution (Art 129) determined by having m for a double ray and the parallel diameter for central ray The two straight lines so drawn will be conjugate to one another with respect to the conic, if then the points where they cut the asymptotes be joined to one another, we shall have two tangents to the curve

382. Let B and C be any two points on a parabola, and A the point where the curve is cut by the diameter which bisects the chord BC Let F and G be two points lying on this diameter which are conjugate with respect to the parabola, *i e* two points equidistant from A (Art. 142), by the theorem of Art 275, BF and CG, and likewise BG and CF, will meet on the curve

This enables us *to construct by points a parabola which circumscribes a given triangle ABC and has the straight line joining A to the middle point of BC as a diameter*

Or we may proceed according to the following method

On BC take two points H and H' which shall be conjugate to one another with regard to the parabola, *i e* any two points dividing BC harmonically Since H and H' are collinear with the pole of the diameter passing through A, therefore by the theorem of Art 275, a point on the parabola will be found by constructing the point of intersection of AH with the diameter passing through H', and another will be found as the point where AH' meets the diameter passing through H

383 In the theorem of Art 274 suppose the tangent c to lie at infinity, then we see that

If a and b are two tangents to a parabola, and if from any point on the diameter passing through the point of contact of a there be drawn two straight lines, one passing through the point ab and the other parallel to b, these will be conjugate to one another with regard to the parabola

This enables us *to construct by tangents a parabola, having given two tangents a and t, the point of contact A of one of them a, and the direction of the diameters*

Draw the diameter through A and let it meet t in O, the second tangent t' from O will be the straight line which is harmonically conjugate to t with respect to the diameter OA (the polar of the point at infinity on a) and the parallel through O to a If now two straight lines h and h' be drawn through O which shall be conjugate to one another with regard to the parabola, *i e* two straight lines which are harmonic conjugates with regard to t and t', the parallel to h' drawn from the point ha and the parallel to h drawn from the point $h'a$ will both be tangents to the required parabola

384 If in the theorem of Art 274 the straight line a be supposed to lie at infinity and b and c to be two tangents to a parabola, we obtain the following

The parallels drawn to two tangents to a parabola, from any point on their chord of contact, are conjugate lines with regard to the conic

By another application of the same theorem we deduce a result already proved in Art 178, viz that

If, from a point on the chord of contact of a pair of tangents b and c to a parabola, two straight lines h and h′ be drawn parallel to b and c respectively, the straight line joining the points hc and h′b will be a tangent to the curve *

From this may be deduced a *construction for the tangents to a parabola determined by two tangents and their points of contact.*

385 THEOREM *If a conic cut the sides BC, CA, AB of a triangle ABC in the points D and D', E and E', F and F' respectively, then will*

$$\frac{BD\ BD'}{CD\ CD'}\ \frac{CE\ CE'}{AE\ AE'}\ \frac{AF\ AF'}{BF\ BF'} = 1 \tag{1}$$

This celebrated theorem is due to CARNOT †

Consider the sides of the triangle ABC (Fig 231) as

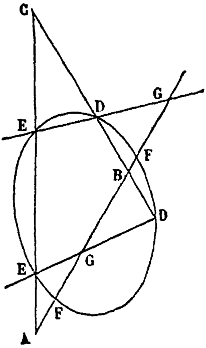


Fig 231

cut by the transversals DE and $D'E'$ in the points D and D', E and E', G and G', by the theorem of Menelaus (Art 139)

$$\frac{BD}{CD}\ \frac{CE}{AE}\ \frac{AG}{BG} = 1, \tag{2}$$

and

$$\frac{BD'}{CD'}\ \frac{CE'}{AE'}\ \frac{AG'}{BG'} = 1 \tag{3}$$

Again, $DEE'D'$ is a quadrangle inscribed in the conic and by Desargues theorem (Art 183) the transversal AB meets the opposite sides and the conic in three pairs of points in involution, therefore (Art 130) the anharmonic ratios $(ABFG)$ and $(BAF'G')$ are equal, thus (Art 45) $(ABFG) = (ABG'F')$, or $(ABFG)\ (ABG'F') = 1$, which gives

$$\frac{AF\ AF'}{BF\ BF'}\ \frac{AG\ AG'}{BG\ BG'} = 1 \tag{4}$$

* DE LA HIRE, *loc cit*, lib III prop 21 † *Géométrie de position*, p 437

Multiplying together (2), (3), and (4), we obtain the relation stated in the enunciation.*

386. Conversely, *if on the sides BC, CA, AB respectively of a triangle ABC there be taken three pairs of points D and D', E and E', F and F' such that the segments determined by them and the vertices of the triangle satisfy the relation* (1) *of Art* 385, *these six points lie on a conic.*

For let the conic be drawn which passes through the five points D, D', E, E', F, and let F'' be the point where it cuts AB again. By Carnot's theorem a relation holds which differs only from (1) in that it has F'' in the place of F'. This relation, combined with (1), gives

$$AF' \cdot BF' = AF'' \cdot BF'',$$

whence $$(ABF'F'') = 1,$$

and therefore (Art. 72, VII) F'' coincides with F'.

* CARNOT'S theorem, being evidently true for the circle (since in this case $BD \cdot BD' = CD \cdot CD'$, &c.), may be proved without making use of involution properties as follows.

Let I, J, K be the points at infinity on BC, CA, AB respectively, and suppose Fig. 231 to have been derived by projecting from any vertex on any plane a triangle $A_1B_1C_1$ whose sides are cut by a circle in D_1 and D_1', E_1 and E_1', F_1 and F_1' respectively. Let I_1, J_1, K_1 be the points on the sides B_1C_1, C_1A_1, A_1B_1 which project into I, J, K respectively; they will of course be collinear. Then

$$\begin{aligned} \frac{BD}{CD} &= (BCDI) \quad \text{(Art. 64)} \\ &= (B_1C_1D_1I_1) \quad \text{(Art. 63)} \\ &= \frac{B_1D_1}{C_1D_1} \cdot \frac{B_1I_1}{C_1I_1}. \end{aligned}$$

So $$\frac{BD'}{CD'} = \frac{B_1D_1'}{C_1D_1'} \cdot \frac{B_1I_1}{C_1I_1}.$$

$$\begin{aligned} \frac{BD \cdot BD'}{CD \cdot CD'} &= \frac{B_1D_1 \cdot B_1D_1'}{C_1D_1 \cdot C_1D_1'} \cdot \frac{B_1I_1^2}{C_1I_1^2} \\ &= \frac{C_1I_1^2}{B_1I_1^2} \quad \text{(Euc. III. 35, 36.)} \end{aligned}$$

Similarly, $$\frac{CE \cdot CE'}{AE \cdot AE'} = \frac{A_1J_1^2}{C_1J_1^2},$$

and $$\frac{AF \cdot AF'}{BF \cdot BF'} = \frac{B_1K_1^2}{A_1K_1^2}.$$

Multiplying these three equations together, and remembering that by the theorem of Menelaus the product on the right hand side is equal to unity, we have the result required.

Carnot's theorem is true not only for a triangle but for a polygon of any number of sides; the proof just given can clearly be extended so as to show this, the theorem of Menelaus being capable of extension to the case of a polygon.

Menelaus' theorem is included in that of Carnot. It is what the latter reduces to when the conic degenerates into two straight lines of which one lies at infinity.

387 If the point A pass off to infinity (Fig 232) the ratios $AF : AE$ and $AF' : AE'$ become in the limit each equal to unity, and the equation (1) of Art 385 accordingly reduces to

$$\frac{BD \cdot BD'}{CD \cdot CD'} \cdot \frac{CE \cdot CE'}{BF \cdot BF'} = 1$$

Draw parallel to BC a straight line to cut CEE' in Q the conic in P and P', the preceding equation, applied to triangle whose vertices are C, Q, and the point at infinity where PP' and BC meet, gives

$$\frac{QE \cdot QE'}{CE \cdot CE'} \cdot \frac{CD \cdot CD'}{QP \cdot QP'} = 1$$

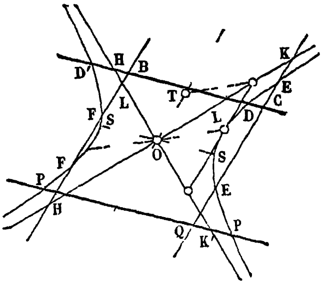


Fig 232

Multiplying together these last two equations, we obtain

$$\frac{BD \cdot BD'}{BF \cdot BF'} = \frac{QP \cdot QP'}{QE \cdot QE'},$$

that is to say

If through any point Q there be drawn in given directions two transversals to cut a conic in P, P' and E, E' respectively, then the rectangles $QP \cdot QP'$ and $QE \cdot QE'$ are to one another in a constant ratio *†

* APOLLONIUS *loc cit*, lib iii 16–23, DESARGUES, *loc cit*, p 202, DE LA HIRE *loc cit* bk v props 10, 12

† From this follows at once the result already proved in a different manner in Art 316, viz that *if a conic is cut by a circle, the chords of intersection make equal angles with the axe*

For let P, P', E, E' be the points of intersection of a circle with the conic, then (Euc iii 35) $QP \cdot QP' = QE \cdot QE'$ But if MCM' NCN' be the diameters of the conic parallel respectively to QPP' and QEE', we have, by the theorem in the text,

$$QP \cdot QP' : QE \cdot QE' = CM \cdot CM' : CN \cdot CN'$$
$$= CM^2 : CN^2$$

Therefore $CM = CN$, and consequently CM and CN (and therefore also QPP' and QEE') make equal angles with the axes

388. Suppose in equation (5) of Art. 387 that the conic is a hyperbola and that in place of BC is taken an asymptote HK of the curve, then the ratio $HD\ HD'$ $KD\ KD'$ becomes equal to unity, and therefore

$$HF\ HF' = KE\ KE',$$

that is to say

If through any point H (or H') lying on an asymptote there be drawn, parallel to a given straight line, a transversal to cut a hyperbola in two points F and F' (D and D'), then the rectangle $HF\ HF'$ ($H'D\ H'D'$) contained by the intercepts will be constant

If the diameter parallel to the given direction $H'D$ meets the curve, then if S and S' are the points where it meets it, and if O is the centre,

$$H'D\ H'D' = OS\ OS' = -OS^2$$

If the diameter OT parallel to the given direction HF does not meet the curve, a tangent can be drawn which shall be parallel to it The square on the portion of this tangent intercepted between its point of contact and the asymptote will be equal to the rectangle $HF\ HF'$ by the theorem now under consideration, but this portion is (Art 303) equal to the parallel semidiameter OT, therefore $HF\ HF' = OT^2$, or

If a transversal cut a hyperbola in F and F' (in D and D') and an asymptote in H (in H'), the rectangle $HF\ HF'$ ($H'D\ H'D'$) is equal to $\pm$ the square on the parallel semidiameter OT (OS), the positive or negative sign being taken according as the curve has or has not tangents parallel to the transversal

389 If the transversal cuts the other asymptote in L (in L') then by Art 193

$$HF' = FL \text{ or } H'D' = DL',$$

and consequently

$$FH\ FL = -OT^2 \text{ or } DH'\ DL' = OS^2,$$

therefore

If a transversal drawn from any point F (D) on a [illegible], cut the asymptotes in H and L (in H' and L'), the rectangle $FH\ FL$ ($DH'\ DL'$) is equal to $\mp$ the square on the parallel semidiameter, the negative or positive sign being taken according as the curve has or has not tangents parallel to the transversal

390 From the proposition of the last Article may be deduced a *construction for the axes of a hyperbola, having given a pair of conjugate semidiameters OF and OT in magnitude and direction* (Fig. 233).

We first construct the asymptotes. Of the two given semidiameters, let OF be the one which cuts the curve Draw through F a parallel to OT, this will be the tangent at F Take on this parallel FP and FQ each equal to OT, then OP and OQ will be the asymptotes (Art. 304) In order now to obtain the directions of the axes, we have only to find the bisectors of the angle included by the asymptotes, or, in other words, the two perpendicular rays OX, OY which are conjugate to one another in the involution of which OP and OQ are the double rays (Arts 296, 297)

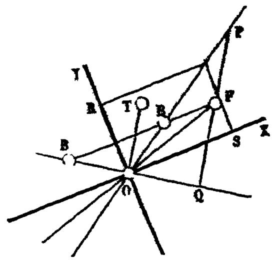


Fig 233.

To determine the lengths of the axes, draw through F a parallel to OX, and let it cut the asymptotes in B and B', and on OX take OS the mean proportional between FB and FB' Then will OS be the length of the semiaxis in the direction OX, and OX will or will not cut the curve according as the segments FB, FB' have or have not the same direction Again, construct the parallelogram of which OS is one side, which has an adjacent side along OY, and one diagonal along an asymptote, its side OR will be the length of the semiaxis in the direction OY (Art 304)

391 In the plane of a triangle ABC take any two points O and O', if OA, OB, OC meet the respectively opposite sides BC, CA AB of the triangle in D, E, F, Ceva's theorem (Art 137) gives

$$\frac{BD}{CD}\,\frac{CE}{AE}\,\frac{AF}{BF} = -1$$

Similarly, if $O'A$, $O'B$, $O'C$ meet the respectively opposite sides in D', E', F', then

$$\frac{BD'}{CD'}\,\frac{CE'}{AE'}\,\frac{AF'}{BF'} = -1$$

If these equations be multiplied together, equation (1) of Art 385 is obtained, therefore

If from any two points the vertices of a triangle are projected upon the respectively opposite sides, the six points so obtained lie on a conic

For example, the middle points of the sides of a triangle and the feet of the perpendiculars from the vertices on the opposite sides are six points on a conic *

* This conic is a circle (the nine point circle) See STEINER *Annales de Mathématiques* (Montpellier, 1828), vol xix p 42, or his Collected Works vol i p 19[illegible]

392. Problem. *To construct a conic which shall pass through three given points A, B, C, and with regard to which the pairs of corresponding points of an involution lying on a given straight line u shall be conjugate points*

Let AB and AC (Fig 234) be joined, and let them meet u in D and E Let the points corresponding in the involution to D and E respectively be D' and E', let D'' be the harmonic conjugate of D

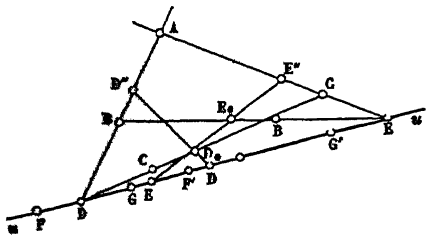
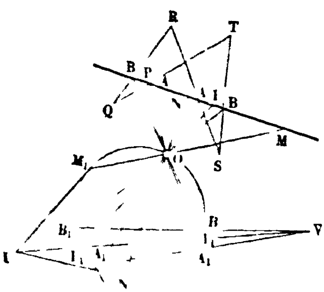


Fig 234.

with respect to A and B, and let E'' be the harmonic conjugate of E with respect to A and C Thus D will be conjugate (with respect to the required conic) both to D' and to D'', and therefore $D'D''$ will be the polar of D So too $E'E''$ will be the polar of E

Join BE, CD, and let them cut $E'E''$ and $D'D''$ in E_0 and D_0 respectively, then E_0 will be conjugate to E and D_0 to D If then two points B', C' be found such that the ranges $BB'EE_0$ and $CC'DD_0$ are harmonic, they will both belong to the required conic

In the figure, F and F', G and G' are the pairs of points which determine on u the involution of conjugate points



393 Problem *To construct a conic which shall pass through four given points Q R, S, T and shall divide harmonically a given segment MN* (Fig 235)

Let the pairs of opposite sides of the quadrangle $QRST$ meet the straight line MN in A and A', B and B' If the required conic cuts MN the two points of intersection will be a pair of the involution determined by A and A', B and B' (Art 183) If then the involution of which M and N are the double points and the involution determined by the pairs of points A and A', B and B' have a pair P and P' in common the required conic will pass through each of the points P and P' (Arts 125, 208)

In order to construct these points, describe any circle (Art. 208) and from any point O on it project the points A, A', B, B', M, N upon the circumference, and let $A_1, A_1', B_1, B_1', M_1, N_1$ be their respective projections. If the chords A_1A_1' and B_1B_1' meet in V, and the tangents at M_1 and N_1 meet in U, all straight lines passing through U determine on the circumference, and consequently (by projection from O) on the straight line MN, pairs of conjugate points of the first involution, and the same is true, with regard to the second involution, of straight lines passing through V. If the straight line UV meets the circle in two points P_1 and P_1', let these be joined to O, the joining lines will cut MN in the required points P and P'.

Let W be the pole of UV with respect to the circle. Every straight line passing through W and cutting the circle deter[mines] on it two points which are harmonically conjugate with regar[d to] P_1 and P_1', and these points, when projected from O on MN, will give two points which are harmonically conjugate with regard to P and P', and which are therefore conjugate to one another with respect to the required conic. If then UV does not cut the circle, so that the points P and P' cannot be constructed, draw through W two straight lines cutting the circle, and project the points of intersection from the centre O upon the straight line MN, this will give two pairs of points which will determine the involution on MN of conjugate points with respect to the conic. The problem therefore reduces to that treated of in the preceding Article.

394 PROBLEM. *To construct a conic which shall pass through four given points Q, R, S, T, and through two conjugate points (which are not given) of a known involution lying on a straight line u.*

This problem is similar to the preceding one, since it amounts to constructing the pair of conjugate points common to the given involution and to that determined on u by the pairs of opposite sides of the quadrangle $QRST$ (Art. 183).

Such a common pair will always exist when the given involution has no double points, and the two points composing it will both lie on the required conic. If the given involution has two double points M and N, the present problem becomes identical with that of Art. 393.

The problem clearly admits of only one solution and the same is the case with regard to those of the two preceding Articles.

395 Consider a hyperbola whose asymptotes are perpendicular to one another, and to which on this account is given the name of *rectangular* hyperbola (Fig. 236). Since the asymptotes are harmonically conjugate with regard to any pair of conjugate diameters (Art. 296), they will in

this case be the bisectors of the angle included between any such pair (Art. 60). But the parallelogram described on two conjugate semidiameters as adjacent sides has its diagonals parallel to the asymptotes (Art. 304), in this case therefore every such parallelogram is a rhombus, that is, *every diameter is equal in length to its conjugate*. On account of this property the rectangular hyperbola is also called *equilateral*.*

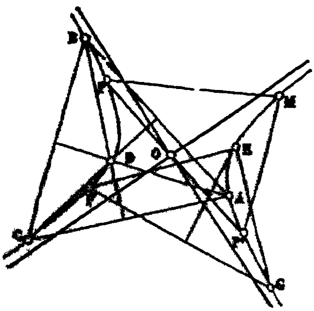


Fig 236

I. Since the chords joining the extremities P and P' of any diameter to any point M on the curve are parallel to a pair of conjugate diameters (Art. 287), the angles made by PM and $P'M$ with either asymptote are equal in magnitude and of opposite sign. If the points P and P' remain fixed, while M moves along the curve, the rays PM and $P'M$ trace out two pencils which are oppositely equal to one another (Art. 106).

II. Conversely, *the locus of the points of intersection of pairs of corresponding rays of two oppositely equal pencils is an equilateral hyperbola*.

For, in the first place, the locus is a conic, since the two pencils are projective (Art. 150). Further, the two pencils have each a pair of rays which include a right angle, and which are parallel respectively to the corresponding rays of the other pencil (Art. 106), the conic has thus two points at infinity lying in directions at right angles to one another and is therefore an equilateral hyperbola. It will be seen moreover that the centres P and P' of the two pencils are the extremities of a diameter. For the tangent p at P is the ray corresponding to PP' regarded as a ray p' of the second pencil, and the tangent q at P' is the ray corresponding to PP' regarded as a ray q of the first pencil (Art. 150), but the angles pq and $p'q'$ must be equal and opposite; therefore, since p' and q coincide, p and q' must be parallel to one another.

III. The angular points of a triangle ABC and its orthocentre (centre of perpendiculars) D are the vertices of a

* Apollonius, *loc cit* vii 21; De la Hire *loc cit*, book v prop 13.

complete quadrangle in which each side is perpendicular to the one opposite to it, and whose six sides determine on the straight line at infinity three pairs of points subtending each a right angle at any arbitrary point S The three pairs of rays formed by joining these points to S belong therefore to an involution in which every ray is perpendicular to its conjugate (Arts 131 left, 124, 207)

But this involution of rays projects from S the involution of points which, in accordance with Desargues' theorem, is determined on the straight line at infinity by the pairs of opposite sides of the quadrangle and by the conics (hyperbolas *) circumscribed about it. The pairs of conjug therefore of the first involution give the directions of asymptotes of these conics, thus

If a conic pass through the angular points of a triangle and through the orthocentre, it must be an equilateral hyperbola †

IV Conversely, if an equilateral hyperbola be drawn to pass through the vertices A, B, C of a triangle, it will pass also through the orthocentre D For imagine another hyperbola which is determined (Art 162, I) by the four points A, B, C, D and by one of the points at infinity on the given hyperbola This new hyperbola will be an equilateral one by the foregoing theorem, and will consequently pass through the second point at infinity on the given curve, and since the two hyperbolas thus have five points in common (A, B, C, and two at infinity) they must be identical, which proves the proposition Therefore

If a triangle be inscribed in an equilateral hyperbola, its orthocentre is a point on the curve

V If the point D approach indefinitely near to A, i e if BAC becomes a right angle, we have the following proposition

If AFG (Fig 236) is a triangle, right-angled at E, which is

* No ellipse or parabola can be circumscribed about the quadrangle here considered (Art 219)

† This may be deduced directly from Pascal's theorem For let a conic be drawn through A B C D and let I_1 and I_2 be the points where it meets the line at infinity Since $ABCDI_1I_2$ is a hexagon inscribed in a conic, the intersections of AB and DI_1, of BC and I_1I_2, and of CD and I_2A are three collinear points Therefore the straight line joining the point in which DI_1 meets AB to that in which AI meets CD must be parallel to BC Thus AI must be at right angles to DI_1, and as these lines are parallel to the asymptotes of the conic the latter is a rectangular hyperbola

inscribed in an equilateral hyperbola, the tangent at E is perpendicular to the hypotenuse FG

VI. Through four given points Q, R, S, T can be drawn only one equilateral hyperbola (Art. 394) The orthocentre of each of the triangles QRS, RST, STQ, QRT lies on the curve *

VII. *Given four tangents to an equilateral hyperbola, to construct the curve*

Since the diagonal triangle of the quadrilateral formed by the four tangents is self-conjugate with respect to the hyperbola, the centre of the latter will lie on the circle circumscribing this triangle (Art. 370, II) But the centre of the hyperbola lies also on the straight line which joins the middle points of the diagonals of the quadrilateral (Art 318, II) Either of the points of intersection of this straight line with the circle will therefore give the centre of an equilateral hyperbola satisfying the problem, there are therefore two solutions For another method of solution see Art. 372

VIII *The polar reciprocal of any conic with respect to a circle* K *having its centre on the director circle is an equilateral hyperbola*

For since the tangents to the conic from the centre O of the circle K are mutually perpendicular, the conic which is the polar reciprocal of the given one must cut the straight line at infinity in two points subtending a right angle at O That is to say, it must be an equilateral hyperbola

396 Suppose given a conic, a point S, and its polar s, and let a straight line passing through S cut the conic in A and A' Let the figure be constructed which is homological with the given conic, S being taken as centre of homology, s as axis of homology, and A, A' as a pair of corresponding points Then every other point B' which corresponds to a point B on the conic will lie on the conic itself For if AB meets the axis s in P then B', the point of intersection of SB and $A'P$ is likewise a point on the conic (Art 250) The curve homological with the given conic will therefore be the conic itself Any two corresponding points (or straight lines) are separated harmonically by S and s this is in fact, the case of *harmonic homology* (Arts 76 298)

To the straight line at infinity will therefore correspond the

* These theorems are due to BRIANCHON and PONCELET, they were enunciated in a memoir published in vol xi of the *Annales de Mathématiques* (Montpellier, 1821) and were given again in vol ii (p 504) of PONCELET's *Applications d'Analyse et de Géométrie* Paris 1864

straight line j which is parallel to s and which lies midway between S and s, and the points in which j meets the conic will correspond to the points at infinity on the same conic.

From this may be derived a very simple *method of determining whether a given arc of a conic, however small, belongs to an ellipse, a parabola, or a hyperbola.*

Draw a chord s joining any two points in the arc, construct its pole S, and draw a straight line j parallel to s and equidistant from S and s. If j does not cut the arc, the latter is part of an ellipse (Fig. 237 *a*). If j touches the arc at a point J, the arc belongs to a parabola of which SJ is a diameter (Fig. 237 *b*). If, finally, j cuts the arc in two points J_1, J_2 (Fig. 237 *c*), the arc will be part of a hyperbola whose asymptotes are parallel to SJ_1 and SJ_2.*

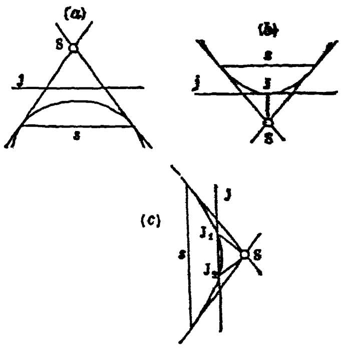


Fig. 237.

397. PROBLEM. *Given a tangent to a conic, its point of contact, and the position (but not the magnitude) of a pair of conjugate diameters, to construct the conic* (Fig. 238).

Suppose O the point of intersection of the given diameters, and P and Q the points in which they are cut by the given tangent. Through the point of contact M of this tangent draw parallels to OQ, OP to meet OP, OQ in P' and Q' respectively. Since the polar of M (the tangent) passes through P, the polar of P will pass

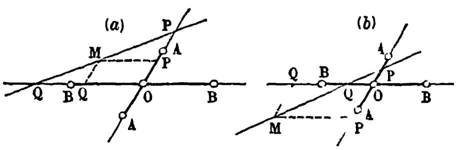


Fig. 238.

through M, and since the polar of P is parallel to OQ, it must be MP'; therefore P and P' are conjugate points.

If now points A and A' be taken on OP such that OA and OA' may each be equal to the mean proportional between OP and OP', then AA' will be equal in length to the diameter in the direction OP (Art. 290). In the same way the length of the other diameter BB' will be found by making OB and OB' each equal to the mean proportional between OQ and OQ'.

* PONCELET, *loc. cit.*, Arts. 225, 226.

If the points P and P' fall on the same side of O, the involution of conjugate points has a pair of double points A and A' (Art. 128), that is to say, the diameter OP meets the curve. If, on the other hand, P and P' lie on opposite sides of O, the involution has no double points, and the diameter OP does not meet the curve. In this case A and A' are two conjugate points lying at equal distances from O. The figure shows two cases: that of the ellipse (a) and that of the hyperbola (b).

298. Problem. *Given a point M on a conic and the positions of two pairs of conjugate diameters a and a', b and b', to construct the conic.*

I. *First solution* (Fig. 239). Through M draw chords parallel to each diameter, and such that their middle points lie on the respectively conjugate diameters. The other extremities A, A', B, B' of

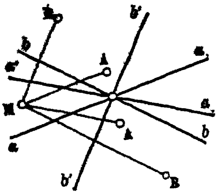


Fig. 239.

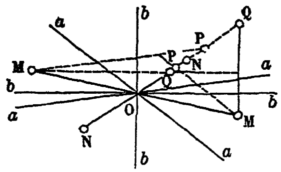


Fig. 240.

the four chords so drawn will be four points all of which lie on the required conic.

II. *Second solution* (Fig. 240). Denoting the diameter MOM' by c, if the ray c' be constructed which is conjugate to c in the involution determined by the pairs of rays a and a', b and b', then c' will be the diameter conjugate to c (Art. 296). Through M draw MP parallel to a, and through M' draw $M'P'$ parallel to a'; these parallels will intersect on the conic (Art. 288); let them cut c' in P and P' respectively. These last two points are conjugate with respect to the conic (Art. 299); thus if on c' two other points be found which correspond to one another in the involution determined by the pair P, P' and the central point O, then MQ and $M'Q'$ will intersect on the conic. If then on c' two points N and N' be taken such that the distance of either of them from O is a mean proportional between OP and OP', they will be the extremities of the diameter c' (Art. 290).

III. *Third solution*. Through the extremities M and M' of the diameter which passes through the given point draw parallels to a and a'; they will meet in a point A lying on the conic. Through the same points draw parallels to b and b'; these will meet in another point B also lying on the conic (Art. 288). Produce AO to A',

making OA' equal to AO, and similarly BO to B', making OB' equal to BO, then will A' and B' be points also lying on the required conic (Art. 281)

399 PROBLEM *Given in position two pairs of conjugate diameters a and a', b and b' of a conic, and a tangent t, to construct the conic.*

I. *First solution* (Fig 241) Let O be the point of intersection of the given diameters, that is, the centre of the conic Draw parallel to t and at a distance from O equal to that at which t lies, a straight line t', this will be the tangent parallel to t Let the points of intersection of t and t' with a and a' be joined, this will give two other parallel tangents u and u' (Art 288) Another pair of parallel tangents v and v' will be obtained by joining the points where t and t' meet b and b'

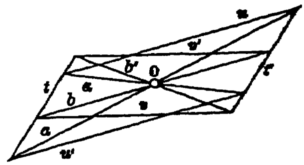


Fig 241

II *Second solution* The conjugate diameters a and a', b and b', will meet t in two pairs of points A and A', B and B' which determine an involution whose centre is the point of contact of t (Art 302) The problem therefore reduces to one already solved (Art. 397) If the involution has double points, the straight lines joining these points to O will be the asymptotes

400 PROBLEM *Given two points M and N on a conic and the position of a pair of conjugate diameters a and a', to construct the conic* (Fig 242)

Let M' and N' be the other extremities of the diameters passing through M and N Through M and M' draw MH, $M'H$ parallel to a and a' respectively, similarly, through N and N' draw NK, $N'K$ parallel to a and a' respectively The points H and K will lie on the required conic

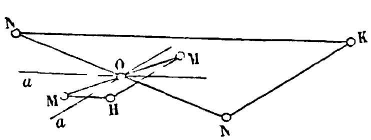


Fig 242

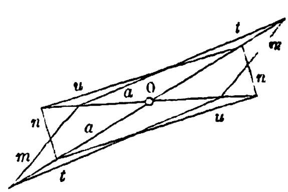


Fig 243

401 PROBLEM *Given two tangents m and n to a conic and the position of a pair of conjugate diameters a and a', to construct the conic* (Fig 243)

Draw the straight lines m' and n' parallel respectively to m and n, and at distances from the centre O equal respectively to those at which m and n lie, then m' will be the tangent parallel to m and n'

the tangent parallel to n. Join the points where m and m' meet a and a' by the straight lines t and t', and the points where n and n' meet a and a' by the straight lines u and u'. The four straight lines t, t', u, u' will all be tangents to the required conic (Art. 288).

402. PROBLEM. *Given five points on a conic, to construct a pair of conjugate diameters which shall make with one another a given angle* *.

Construct first a diameter AA' of the conic (Art. 285), and on it describe a segment of a circle containing an angle equal to the given one. Find the points in which the circle of which this segment is a part cuts the conic again (Art. 227); if M is one of these points, AM and $A'M$ will be parallel to a pair of conjugate diameters. Since then AMA' is equal to the given angle, the problem will be solved by drawing the diameters parallel to AM and $A'M$.

If the segment described is a semicircle, this construction gives the axes.

403. PROBLEM. *To construct a conic with respect to which a given triangle EFG shall be self-conjugate, and a given point P shall be the pole of a given straight line p* †.

Let p meet FG in A. The polar of A will pass through E the pole of FG, and through P the pole of p, and will therefore be EP. Similarly FP, GP will be the polars of the points B, C in which p is cut by GE, EF respectively. Let A' be the point in which FG intersects EP, then F and G, A and A', are two pairs of conjugate points with respect to the conic, and if the involution which they determine has a pair of double points L and L', these points will lie on the required conic (Art. 264). The same construction may be repeated in the case of the other two sides of the triangle EFG.

If the point P lies within the triangle EFG the points A', B', C' lie upon the sides FG, GE, EF respectively (not produced ‡). The straight line p may cut two of the sides of the triangle, or it may lie entirely outside the triangle. In the first case the involutions lying on the two sides of the triangle which are cut by p are both of the non-overlapping (hyperbolic) kind, and therefore each possesses double points (Art. 128); these give four points of the required curve, and the problem reduces to that of describing a conic passing through four given points and with respect to which two other given points are conjugates (Art. 393). In the second case, on the other hand, the pairs of conjugate points on each of the sides of the triangle EFG overlap and the involutions have no double points (Art. 128), in

* DE LA HIRE, *loc. cit.* book ii. prop. 38.

† STAUDT, *Geometrie der Lage*, Art. 237.

‡ We shall say that a point A lies *on* the side FG of the triangle, when it lies between F and G; and that a straight line *cuts* the side FG, when its point of intersection with FG lies between F and G.

this case the conic does not cut any of the sides of the self-conjugate triangle, therefore (Art. 262) it does not exist.

If the point P lies outside the triangle, one only of the three points A', B', C' lies on the corresponding side, the two others lie on the respective sides produced If these two other sides are cut by p, none of the involutions possesses double points, and the conic does not exist If, on the other hand, p cuts the first side, or if p lies entirely outside the triangle, the conic exists, and may be constructed as above

In all cases, whether the conic has a real existence or not, the *polar system* (Art 339) exists It is determined by the self-conjugate triangle EFG, the point P, and the straight line p To construct this system is a problem of the first degree, while the construct the conic is a problem of the second degree

404 PROBLEM *Given a pentagon $ABCDE$, to describe a conic with regard to which each vertex shall be the pole of the opposite side* *

Let F be the point of intersection of AB and CD If the conic **K** be constructed (Art 403) with regard to which ADF is a self-conjugate triangle and E the pole of BC, then the points B and C in which BC is cut by AF and DF respectively will be the poles of ED and EA, the straight lines which join E to the points D and A respectively Every vertex of the pentagon will therefore be the pole of the opposite side, that is, **K** will be the conic required

If the conic **C** be constructed which passes through the five vertices of the pentagon, and also the conic **C**$'$ which touches the five sides of the pentagon (Art 152), these two conics will be polar reciprocals one of the other with respect to **K** (Art 322)

405 PROBLEM *Given five points A, B, C, D, E (no three of which are collinear), to determine a point M such that the pencil $M(ABCDE)$ shall be projective with a given pencil $abcde$* (Fig 244)

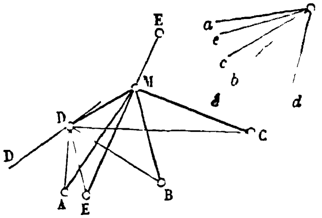


Fig 244

Through D draw two straight lines DD', DE' such that the pencil $D(ABCD'E')$ is projective with $abcde$ (Art 84, right) Construct the point E' in which DE' meets the conic which passes through the four points $ABCD$ and touches DD' at D (Art 165), then construct the point M in which the same conic meets EE' M will be the point required For since M, A, B, C, D, E' lie on the same conic the pencil $M(ABCDE')$ is projective with the pencil $D(ABCD'E')$,

* STAUDT *loc cit*, Arts. 238, 258

which by construction is projective with the given pencil *abcde*. Since then MN' and MN are the same ray, the problem is solved.

As an exercise may be solved the correlative problem, viz.

Given five straight lines a, b, c, d, e, no three of which are concurrent, to draw a straight line m to meet them in five points forming a range projective with a given range $ABCDE$. *

406. Problem. *To trisect a given arc AB of a circle.* †

On the given arc take (Fig. 245) a point N, and from B measure in the opposite direction to AN an arc BN' equal to twice the arc AN. If BT be the tangent at B, and if O be the centre of the circle

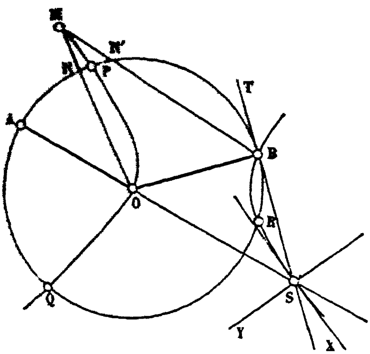


Fig. 245.

of which the arc AB is a part, the angles AON and TBN' are equal and opposite. If N and N' vary their positions simultaneously, the rays ON and BN' will describe two oppositely equal pencils, and the locus of their point of intersection M will therefore (Art. 395, II) be an equilateral hyperbola passing through O and B. The asymptotes of this hyperbola are parallel to the bisectors of the angle made by AO and BT with one another, for these straight lines are corresponding rays (being the positions of the variable rays ON and BN' for which the arcs AN and BN' are each zero). The centre of the hyperbola is the middle point of the straight line OB which joins the centres of the two pencils.

The hyperbola having been constructed by help of Pascal's theorem, the point P will have been found in which it cuts the arc AB. Two corresponding points N and N' coalesce in this point; therefore the arc AP is half of the arc PB, and P is that point of trisection of the arc AB which is the nearer to A.

The hyperbola meets the circle in two other points R and Q. The point R is one of the points of trisection of the arc which together

* STAUDT, *loc. cit.*, Art. 63.

† STAUDT, *Beiträge*, Art. 452; CHASLES, *Sections coniques*, Art. 37.

with AB makes up a semicircle, and the point Q is one of the points of trisection of the arc which together with AB makes up the circumference of the circle

407 It has been seen (Art. 191) that if P', P'', Q', Q'' (Fig 246) are four given collinear points, and if any conic be described to pass through P' and P'', and then a tangent be drawn to this conic from Q' and another from Q'', the chord joining the points of contact of these tangents passes through one of the double points M', N' of the involution which is determined by the two pairs of points P' and P'', Q' and Q'' The two tangents which can be drawn from Q', combined with the two from Q'', give four such chords of contact, of which two pass through M' and two through N' From this may

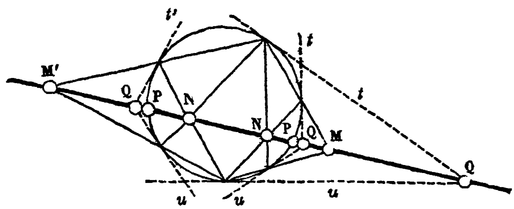


Fig 246

be deduced *a construction for the double points of the involution* $P'P''$, $Q'Q''$, or, what is the same thing (Art 125), *for the two points* M', N' *which divide each of the two given segments* $P'P''$ *and* $Q'Q''$ *harmonically*

Describe any circle to pass through P' and P'', and draw to it from Q' the tangents t' and u', and from Q'' the tangents t'' and u'' The chord of contact of the tangents t' and t'' and that of the tangents u' and u'' will cut the straight line $P'P''$ in the two required points M' and N'

408 This construction has been applied by BRIANCHON * to the solution of the two problems considered in Art 221 viz

I *To construct a conic of which two points* P', P'' *and three tangents* q q', q'' *are given*

Join $P'P''$, and let it cut the three given tangents in Q Q', Q'' respectively (Fig 246) Describe any circle through P', P'' and draw to it tangents from Q, Q', Q'' The chords which join the points of contact of the tangents from Q'' to the points of contact of the tangents from Q meet $P'P''$ in two points M and N and similarly the tangents from Q'' combined with those from Q' determine two points M' and N'

The chord of contact of the tangents q', q'' to the required conic will therefore pass through one of the points M, N, and that of the

* BRIANCHON, *loc cit*, pp 47, 51

tangents q', q'' will pass through one of the points M', N' The four combinations MM', MN', NM', NN' give the four solutions of the problem.

The problem therefore reduces to the following *To describe a conic which shall touch three given straight lines q, q', q'' in such a way that the chords of contact of the two pairs of tangents q, q'' and q', q'' shall pass respectively through two given points M and M'* Let $QQ'Q''$ (Fig 247) denote the triangle formed by the three given tangents, and let A, A', A'' be the points of contact to be determined. By a corollary to Desargues' theorem (Art 194), the side $q \equiv Q'Q''$ is divided harmonically at the point of contact A and at the point where it is cut by the chord $A'A''$ If these four harmonic points be projected on MQ'' from A'' as centre, it follows that the segment RQ'' intercepted on MQ'' between q'' and q' is divided harmonically by M and the chord $A'A''$

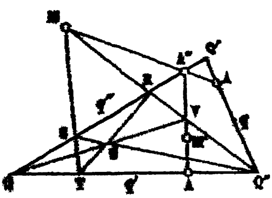


Fig. 247

Let then MQ'' be joined, it will cut q'' in some point R, and let the point V be determined which is harmonically conjugate to M with regard to R and Q'' In order to do this, draw through M any straight line to cut q'' and q' in S and T respectively, join SQ'' and TR, meeting in U, and join QU, meeting RQ'' in V Join VM', it will meet q' and q'' in A' and A'', and finally if MA'' be joined it will cut $Q'Q''$ in A

II *To construct a conic of which three points P, P', P'' and two tangents q, q' are given*

Join PP', and let it meet q and q' in Q and Q' respectively, join PP'', and let it meet q and q' in R and R' respectively Describe a circle round $PP'P''$, and to it draw tangents from Q and Q', the chords of contact will meet PP' in two points M and N Similarly draw the tangents from R and R', the chords of contact will meet PP' in two other points M' and N' Then each of the straight lines MN', NN' $M'N$ MM' will meet the tangents q and q' in two of the points of contact of these two tangents with a conic circumscribing the triangle $PP'P''$

This construction differs from that given in Art 221 (left) only in the method of finding the double points M and N M' and N'

409 THEOREM *If two angles AOS and $AO'S$ of given magnitude turn about their respective vertices O and O' in such a way that the point of intersection S of one pair of arms lies always on a fixed straight line u the point of intersection of the other pair of arms will describe a conic* (Fig 248)

The proof follows at once from the property that the pencils traced out by the variable rays OA and OS, OS and $O'S$, $O'S$ and $O'A$ are projective two and two (Arts. 42, 108), and that consequently the pencils traced out by OA and $O'A$ are projective. This theorem is due to NEWTON, and was given by him under the title of *The Organic Description* of a conic *

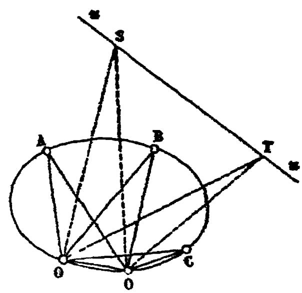


Fig 248

410. The following, which depend on the foregoing theorem, may serve as exercises to the student —

1. Deduce a construction for a conic passing through five given points O, O', A, B, C

2. Given these five points, determine the magnitude of the angles AOS, $AO'S$ and the position of the straight line u in order that the conic generated may pass through the five given points

3. On the straight line OO' which joins the vertices of the two given angles a segment of a circle is described containing an angle equal to the difference between four right angles and the sum of the given angles. Show that according as the circle of which this segment is a part cuts, does not cut, or touches the straight line u, so the conic generated will be a hyperbola, an ellipse, or a parabola.

4. Determine the asymptotes of the conic, supposing it to be a hyperbola, or its axis, in the case where it is a parabola

5. When is the conic (*a*) a circle, (*b*) an equilateral hyperbola, (*c*) a pair of straight lines?

6. Examine the cases in which the two given angles are directly equal, or oppositely equal, or supplementary †

411. THEOREM. *If a variable triangle AMA' move in such a way that its sides turn severally round three given points O, O', S* (Fig 249) *while two of its vertices A, A' slide along two fixed straight lines u, u' respectively, the locus of the third vertex M is a conic passing through the following five points, viz O, O', uu', and the intersections B and C' of u and u' with $O'S$ and OS respectively* ‡

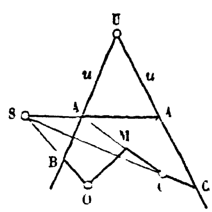


Fig 249

* *Principia* lib I lemma XXI, *Enumeratio linearum tertii ordinis* (*Opticks* 1704) p 158, § XXXI

† MACLAURIN *Geometria Organica* (London, 1720), sect I prop 2

‡ See Art 156

412. THEOREM. (The theorem of Art. 411 is a particular case of this). *If a variable polygon move in such a way that its n sides turn severally round n fixed points $O_1, O_2, \ldots O_n$ (Fig 250) while $n-1$ of its vertices slide respectively along $n-1$ fixed straight lines $u_1, u_2, \ldots u_{n-1}$, then the last vertex will describe a conic, and the locus of the point of intersection of any pair of non-adjacent sides will also be a conic* *

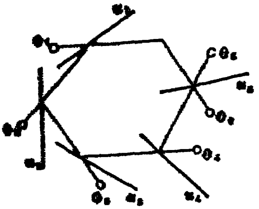


Fig 250.

The proof of this theorem and its correlative is left to the student †

413. THEOREM. *From two given points A and A′ tangents AB, AC and A′B′, A′C′ are drawn to a conic, then will the four points of contact B, C, B′, C′, and the two given points A, A′ all lie on a conic* (Fig. 251 ‡).

Let $A'C'$, $A'B'$ meet BC in D and E respectively, these points will evidently be the poles of AC', AB' respectively. The pencil $A(BCB'C')$ is projective with the range of poles $BCED$ (Art 291), and therefore with the pencil $A'(BCED)$ or $A'(BCB'C')$, which proves the theorem

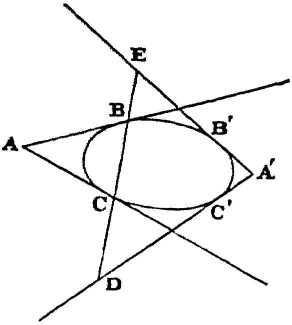


Fig 251

414. THEOREM (correlative to that of Art 413) *From two given points A and A′ tangents AB, AC and A′B′, A′C′ are drawn to a conic, then will the four tangents and the two chords of contact all touch a conic* ‡

For (Fig 251) the range of points BC $(AB, AC, A'B', A'C')$ or $BCED$ is projective with the pencil $A(BC\,B'C')$ formed by their poles, but this pencil is projective with the range $B'C'$ $(AB, AC, A'B', A'C')$, therefore the six lines AB, AC, $A'B'$, $A'C'$, BC, $B'C'$ all touch a conic

415 THEOREM *On each diagonal of a complete quadrilateral is taken a pair of points dividing it harmonically if of these six points three (one from each diagonal) lie in a straight line, the other three will also lie in a straight line*

COROLLARY *The middle points of the three diagonals of a complete quadrilateral are collinear*

* This theorem is due to MACLAURIN and BRAIKENRIDGE (*Phil Trans*, London 1735)

† PONCELET *loc cit* Art 502

‡ CHASLES *Sections coniques*, Arts 213 214

416 THEOREM *If from any point O on the circle circumscribing a triangle ABC straight lines OA', OB', OC' be inflected to meet the sides BC, CA, AB in A', B', C' respectively, and to make with them equal angles (both as regards sign and magnitude), then the three points A', B', C' will be collinear* (Fig 252)

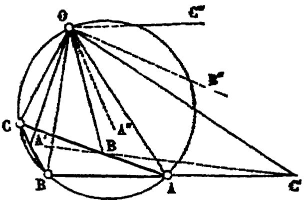


Fig 252

Through O draw OA'', OB'', OC'' parallel to BC, CA, AB respectively, then it is easily seen that the angles AOA'', BOB'', COC'' have the same bisectors The same will therefore be true with regard to the angles AOA', BOB', COC', consequently (Art 142) the arms of these last three angles will form an involution, and therefore (Art 135) the points A', B', C' will be collinear * †

417 THEOREM *If from the vertices of a triangle circumscribed about a circle straight lines be inflected to meet any tangent to the circle, so that the angles they subtend at the centre may be equal (in sign and magnitude), then the three straight lines will meet in a point* ‡

The proof is similar to that of the theorem in the preceding Article

418 PROBLEMS (1) Given three collinear segments AA', BB', CC', to find a point at which they all subtend equal angles (Art 109)

In what case can these angles be right angles? (See Art 128)

(2) Given two projective ranges lying on the same straight line, to find a point which is harmonically conjugate to a given point on the line, with respect to the two self corresponding points of the two ranges (which last two points are not given) §

(3) Given two pairs of points lying on a straight line, to determine on the line a fifth point such that the rectangle contained by its distances from the points of the first pair shall be to that contained

* CHASLES *loc cit*, Art 386

† *Otherwise* Since the triangles BOC, COB are similar,

$$BC \quad CB' = OB \quad OC$$

So also $$CA \quad AC' = OC \quad OA,$$

and $$AB' \quad BA' = OA \quad OB$$

whence by multiplication, paying attention to the signs of the segments

$$BC \quad CA' \quad AB' = -C'A \quad B'C \quad AB,$$

which shows (Art 139) that A, B', C are collinear

‡ CHASLES, *loc cit*, Art 387

§ CHASLES, *Geom sup*, Art 269

by its distances from the points of the second pair in a given ratio *

(4). Through a given point to draw a transversal which shall cut off from two given straight lines two segments (measured from a fixed point on each line) which shall have a given ratio to one another, or, the rectangle contained by which shall be equal to a given one †

419 It will be a useful exercise for the student to apply the theory of pole and polar to the solution of problems of the first and second degree, supposing given a ruler, and a fixed circle and its centre. We give some examples of problems treated in this manner

I. *To draw through a given point P a straight line parallel to a given straight line q*

The pole Q of q and the polar p of P (with respect to the given circle) must be found, if A be the point where p is cut by the straight line OQ joining Q to the centre of the circle, then the polar a of A will be the straight line required

II. *To draw from a given point P a perpendicular to a given straight line q*

Draw through P a straight line parallel to OQ, it will be the perpendicular required

III *To bisect a given segment AB*

Let a and b be the polars of A and B respectively, and c that diameter of the given circle which passes through ab, if d be the harmonic conjugate of c with respect to a and b, the pole of d will be the middle point of AB

IV *To bisect a given arc MN of a circle*

Construct the pole S of the chord MN, the diameter passing through S will cut MN in the middle point of the latter

V *To bisect a given angle*

If from a point on the circle parallels be drawn to the arms of the given angle the problem reduces to the preceding one

VI *Given a segment AC to produce it to B so that AB may be double of AC*

Let a and c be the polars of A and C respectively, d the diameter of the given circle which passes through ac, and b the ray which makes the pencil $abcd$ harmonic, the pole of b will be the required point B

* This is the problem 'de sectione determinata' of APOLLONIUS See CHASLES, *Géom sup*, Art 281

† These are the problems 'de sectione rationis' and 'de sectione spatii' of APOLLONIUS See CHASLES, *Géom sup*, Arts 296, 298

VII. *To construct the circle whose centre is at a given point U and whose radius is equal to a given straight line UA*

Produce AU to B, making UB equal to AU (by VI), and draw perpendiculars at A and B to AB (by II). Bisect the right angles at A and B (by V), and let the bisecting lines meet in C and D. We have then only to construct the conic of which AB and CD are a pair of conjugate diameters (Art 301)

420 The following problems* depend for their solution on the theorem of Art 376

I *Given three points A, B, C on a conic and one focus F, to construct the conic*

With centre F and any radius describe a circle **K**, and let the polars of A, B, C with respect to this circle be a, b, c respectively. Describe a circle touching a, b, c and take its polar reciprocal with respect to **K**, this will be the conic required.

Since there can be drawn four circles touching a, b, c (the inscribed circle of the triangle abc and the three escribed circles), there are four conics which satisfy the problem

II *Given two points A, B on a conic, one tangent t, and a focus F, to construct the conic*

Describe a circle **K** as in the last problem, and let a, b be the polars of A, B, and T the pole of t, with respect to **K**. Draw a circle to pass through T and to touch a and b, the polar reciprocal of this circle with respect to **K** will be the conic required

Since four circles can be drawn to pass through a given point and touch two given straight lines, this problem also admits of four solutions

III *Given one point A on a conic, two tangents b, c, and a focus F, to construct the conic*

Describe a circle **K** as in the last two problems, let a be the polar of A, and let B, C be the poles of b, c respectively with regard to this circle Draw a circle to pass through B and C and to touch a, its polar reciprocal with respect to **K** will be the conic required

Since two circles can be described through two given points to touch a given straight line, this problem admits of two solutions

IV *Given three tangents a, b, c to a conic and one focus F to construct the conic*

Describe a circle **K** as in the last three problems, and let A, B, C be the poles of a, b, c respectively with regard to this circle Draw the circle through A, B, C and take its polar reciprocal with respect to **K**, this will be the conic required

This problem clearly admits of only one solution

* Solutions of these problems were given by DE LA HIRE (see CHASLES, *Aperçu historique*, p 125), and by NEWTON (*Principia*, lib i. props 19 20, 21)

431. PROBLEM. *Given the axes of a conic in position (not in magnitude) and a pair of conjugate straight lines which cut one another orthogonally, to construct the foci.*

If O be the centre of the conic, and P, P' and Q, Q' the points in which the two conjugate lines respectively cut the axes, then of the two products $OP \cdot OP'$ and $OQ \cdot OQ'$, one will be positive and the other negative. This determines which of the two given axes is the one containing the foci. If now a circle be circumscribed about the triangle formed by the two given conjugate lines and the non-focal axis, it will cut the focal axis at the foci (Art. 343).

432. The following are left as exercises to the student.

1. Given the axes of a conic in position, and also a tangent and its point of contact, construct the foci, and determine the lengths of the axes (Art. 344).

2. Given the focal axis of a conic, the vertices, and one tangent, construct the foci (Art. 360).

3. Given the tangent at the vertex of a parabola, and two other tangents, find the focus (Art. 358).

4. Given the axis of a parabola, and a tangent and its point of contact, find the focus (Art. 346).

5. Given the axis and the focus of a parabola, and one tangent, construct the parabola by tangents (Arts. 346, 349, 358).

6. The locus of the pole of a given straight line r with respect to any conic having its foci at two given points is a straight line r' perpendicular to r. The two lines r, r' are harmonically separated by the two foci.

7. The locus of the centre of a circle touching two given circles consists of two conics having the centres of the given circles for foci.

8. The locus of a point whose distance from a given straight line is equal to its tangential distance from a given circle consists of two parabolas.

9. In a central conic any focal chord is proportional to the square of the parallel diameter.

10. In a parabola twice the distance of any focal chord from its pole is a mean proportional between the chord and the parameter.

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